Theoretical perspectives for the study of contamination between physical and virtual teaching/learning environments

Prospettive teoriche per lo studio della contaminazione tra ambienti di insegnamento/apprendimento fisici e virtuali

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Abstract / We propose an analysis of the effects on the teaching/learning practices of mathematics that stem from digital technologies in school classrooms. We describe two episodes - an exchange of messages in the Stream of Google Classroom and an extract from a private Whatsapp chat between two classmates. We analyze such episodes with a theoretical lens based on the Chevallard Triangle of Didactics (Chevallard & Joshua, 1982) and the sociological perspective, proposed by D'Amore (2005), considering the classroom as a society. Digital technologies enable new complex teaching/learning environments whose study require a systemic and relational approach as the one offered by the concept of Triangle. Moreover, we show how such environments in turn can be considered societies with their constitutive practices and extra-functional meta-practices that are linked with the societies that originate in the school classrooms. We show how the interaction between these theoretical perspectives allows us to characterize teaching/learning processes when physical and virtual environments contaminate each other.

Keywords: Chevallard Triangle; virtual classroom; society; digital technologies; practices and meta-practices.

Sunto / Proponiamo un'analisi degli effetti della presenza delle tecnologie digitali sulle dinamiche di insegnamento/apprendimento nelle pratiche didattiche nelle aule scolastiche. Descriviamo due episodi - uno scambio di messaggi nello Stream di Google Classroom e un estratto di una chat privata su Whatsapp tra due compagni di classe. Analizziamo tali episodi con una lente teorica composta da il "Triangolo della didattica" (Chevallard & Joshua, 1982), e la prospettiva sociologica, proposta da D'Amore (2005), che considera la classe come società. Le tecnologie digitali abilitano nuovi e complessi ambienti di insegnamento/apprendimento il cui studio richiede una visione sistemico relazionale come quella offerta dal Triangolo. Inoltre, si mostra come tali ambienti possono essere considerati a loro volta delle società con le proprie pratiche costitutive e meta-pratiche extrafunzionali che sono in relazione con le società nelle aule scolastiche. Mostriamo come l'interazione tra queste prospettive teoriche permette di caratterizzare i processi di insegnamento/apprendimento quando ambienti fisici e virtuali si contaminano a vicenda.

Parole chiave: Triangolo di Chevallard; classe virtuale; società; tecnologie digitali, pratiche e meta-pratiche

1 Foreword

In order to improve the effectiveness of our message and to avoid possible misunderstandings, we wish to clarify some preliminary issues.

– *Context.* The socio-cultural context we refer to is the Italian one and, from an institutional/educational perspective, we are dealing with upper secondary

school level[1] (grade 9 - 13). This means that whenever we use educational terms such as "class, classroom or law", we are referring to their meaning in this specific context, aware of the obvious limits posed by generic terms. 1

– *Terminology.* In order to avoid the risks of the *object metaphor* as described by Sfard (2008), we consider it necessary to state exactly what we mean by the expression "digital technologies", and outline the teaching/learning (T/L) environment we are referring to. We use the expression "digital technologies" to refer to every feasible combination of hardware and software. For instance, if a person is using a browser on a PC, that person is using a specific example of digital technology; a classroom where a PC is connected to a projector is another environment with a specific example of digital technology.
  We will consider two different T/L environments: physical and virtual.
  The *physical T/L environment* is defined as the physical space where one or more individuals in the role of student and one or more individuals in the role of teacher are interacting in a T/L dynamic. An example of this kind of environment is the classroom, where students participate in the teacher's lesson.
  The *virtual T/L environment* is delineated by the virtual space and exists when one or more individuals in the role of student, and one or more individuals acting as teacher, interact in a T/L dynamic mediated by instances of digital technology. People can interact with one another and be in a relationship both in the first kind of environment and in the second. When it occurs that the same people are involved in a T/L relationship both in a physical and a virtual environment, a new hybrid environment emerges, which is generated by the relation and contamination of the two settings.
– *Distinction and relationship between individuals and accounts*. It is important to underline a crucial distinction: individuals in the physical environment are physical people (PP); individuals in the virtual environment are "accounts" (A). Furthermore, the correspondence between physical people and accounts is not necessarily bijective. In fact, situations may arise where for each physical person there is a single account (#PP = #A), situations where each physical person manages more than one account (#PP < #A) or situations where the same account is managed by more than a physical person (#PP > #A). Naturally, in a T/L context, physical people can implement an aware and strategic use of accounts in order to carry out some operations which have educational relevance. As an example: it could be possible to achieve different levels of monitoring by taking advantage of varying levels of authorisation regarding the actions of accounts; a physical person can decide to hold and separate different roles, which he/she then employs during his/her interactions with other physical people in the same context, and he/she can do so by using different accounts, where each one is specific for each role etc.
– *Call for dialogue*. In this paper, we tackle a theme of great complexity with a significant amount of relevant literature, thus bias and incompleteness may be impossible to eradicate. In light of this fact, we do not consider our contribution in terms of destination but as a starting point which can be enriched by readers' impressions, reflections and experiences. Therefore, we intend to start a communication channel, which can accompany and complement this paper, where

1. The upper secondary school in Italy lasts five years and corresponds to the grades from 9 to 13.

we invite readers to share their reflections, comments, suggestions and criticisms in order to create a type of ongoing appendix. The channel we propose is an online form, created and managed by the two authors of this paper. The form can be accessed by clicking on the following link: https://forms.gle/oEYaSEEo55pouDGX8.

2 Introduction

Current literature in mathematics education offers various theoretical tools which allow us to study and characterise teaching/learning (T/L) dynamics. Facets and modalities of the T/L dynamics depend on the specific context in which they occur.
When digital technologies are introduced into an education system, they have an impact on fundamental and load-bearing elements which span from instrumental to strategic, from personal to social, from pragmatic to affective.
Bearing in mind a traditional mathematics class, the first elements which come to mind are the classroom where the mathematics lesson takes place, the teacher, and the students. Borba, Askar, Engelbrecht, Gadanidis, Llinares and Aguilar (2016) highlight how digital technologies have deconstructed this notion in depth, enabling new forms of classrooms and new ways of working, triggering new socio-cultural dynamics:

> «Currently it seems clear that digital technology is "deconstructing" the notion of the classroom. [...] Mobile technology, PLN[2]s, digital learning objects and other artifacts are "stretching" the classroom, transforming the classroom to the extent that it can hardly be recognized as such. [...] In this scenario, the regular classroom no longer serves as locus for education. Couches, chairs, tables at students' house, cafè and Lan Houses are the new classrooms. Flipped classrooms change the notion of what is in and outside of the classroom and also change the roles of students and teachers».
>
> (Borba et al., 2016, pp. 605-606).

Therefore, where and how this impacts must be described and framed in a specific and precise way; many researchers are contributing to the literature in mathematics education regarding this issue.
Borba et al. (2016) outline five different research streams concerned with the use of digital technologies in mathematics education:

1. The first regards the use of mobile technologies;
2. The second regards MOOCs (Massive Online Open Courses) which are, as the name suggests, online courses open to any number of participants, without demanding any prerequisite;
3. The third regards collections of didactic resources and learning objects, i.e. «[...] any entity, digital or non-digital, that may be used for learning, education or training» (Risk, 2002, p. 6);

2. Personal Learning Network.

4. The fourth regards the use of technologies in collaborative learning;
5. The fifth regards blended courses, which combine online T/L environments with face-to-face experiences and are a solution used in professional development courses for teachers.

We embrace the challenge proposed by Borba et al. (2016) and we provide a theoretical framework to interpret the complexity of these new T/L spaces that originate from the intertwining of physical and virtual classrooms.
We look at possible forms employed by the triadic teacher-pupil-Knowledge system (t-p-K), studying its effects on T/L processes. We focus on interactions between physical and virtual classrooms, using a systemic-relational perspective in order to highlight how students learn and which conditions may activate or inhibit learning.
An additional branch of analysis – which is worthy of note and complementary to our choice – regards the role that digital technologies assumes in delineating distinctive characteristics of cognition and learning in mathematics as well as their processes.
This study follows in the footsteps of the Free University of Bozen (Libera Università di Bolzano)'s research project VirMath and can be positioned within the first and fourth points of the above list. We focus our investigation on pre-university school level, and we propose an analysis of an educational system of T/L of mathematics where physical environments contaminate (and are contaminated by) virtual contexts.

Drijvers, Ball, Barzel, Heid, Cao and Maschietto (2016) presented a research survey on the use of digital technologies at pre-university school level. With regard to technologies for communication, the Authors highlight a distinction between the communication *of* technology and communication *through* technology. Communication of technology means the communication *of* technological output. Communication *through* technology refers both to the display technologies (which allow us to project, show or share mathematical ideas) and to technologies for sharing and collaboration (which allow people who use them to exchange ideas and materials).
In this paper we analyse two examples of communication *through* technology in mathematics in two different and specific contexts, each of which represents a very common situation: the use of a didactic platform and a private chat between students.
In sec. 3, we introduce the theoretical tools with which we frame these contexts of communication and learning characterised and enabled by the use of digital technologies. In sec. 4, we analyse and interpret two examples of implementation of a learning context in the presence of digital technologies, and we show the impact that this kind of technology has on T/L in mathematics. In sec. 5, we conclude with some general and theoretical reflections.

3 Theoretical Framework

In order to take into account the complexity of the didactic system, our theoretical framework is twofold:
– We start by considering systemic models that allow us to frame the individuals

and relations that characterise a didactic context: the classic Triangle of Didactics introduced by Chevallard and Joshua (1982), which generalizes T/L situations in mathematics, accompanied by the Tetrahedron developed by Albano, Faggiano and Mammana (2013) specific to e-learning (henceforth: Tetrahedron).

– Adopting the sociological perspective of the classroom conceived as society as proposed by D'Amore (2005), we will single out some structural variables of the specific didactical context under investigation and that emerge as relevant.

3.1 Systemic models of didactic relationships

Chevallard and Joshua (1982) use the so-called Triangle of Didactics to frame the three pillars that characterise a didactic context: teacher, pupil and Knowledge.

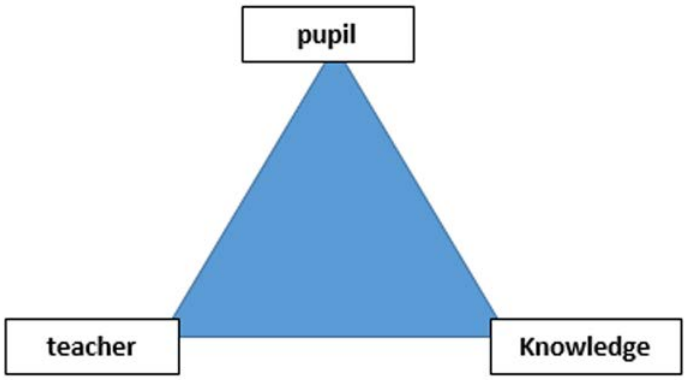

Figure 1
Chevallard's Triangle of Didactics t-p-K.

D'Amore and Fandiño Pinilla (2002) offer a thorough analysis of the triangle and show how its seeming simplicity hides an incredible theoretical power that we will present in sec. 4.

The Authors arrange and articulate the contributions of several researchers to present an analytical reading of the elements that make up such a systemic model. The vertices act as poles and the sides highlight the relations between these poles. In particular, D'Amore and Fandiño Pinilla characterise each pole as an attractor of specific studies: the theory of obstacles, misconceptions, epistemological and ontological aspects of mathematics linked to learning, competences, beliefs and convictions etc. Each side/relation corresponds to activities and theoretical elements: devolution, didactic transposition, and personal implication (Brousseau, 1986, 2002; D'Amore, 1999; D'Amore & Fandiño Pinilla 2002).

In the implementation of the didactic process, Knowledge (intended as academic knowledge) gradually becomes part of the pupil's competent knowledge. Such a path can be broken down in a chain of transformations described by Fandiño Pinilla (2002) as shown in Figure 2:

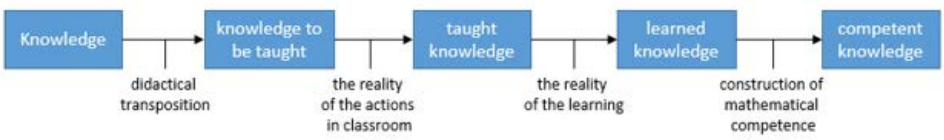

Figure 2
The chain of transformations from Knowledge to competent knowledge (Fandiño Pinilla, 2002, p. 29).

The first transformation, the *didactic transposition*, is the process that transforms academic knowledge into the knowledge to be taught, accessible to the pupil for the construction of mathematical competence (Fandiño Pinilla, 2002, 2003). Factual classroom activity transforms the knowledge to be taught into taught knowledge. The taught knowledge pertains to a student and to his/her situation of learning (which comprehends his/her beliefs, conceptions, previous knowledge etc). The passage from learned knowledge to competent knowledge adds a list of further elements which regard the development of mathematical competence, but this issue goes beyond the purpose of this study, so we refer the reader to Fandiño Pinilla (2003) for further details.
The introduction of digital technologies into the T/L of mathematics unavoidably increases the complexity of the didactic system. Albano (2017) and Albano et al. (2013), propose a Tetrahedron model to frame the specific case of e-learning in mathematics. In this model, the four vertices represent the elements involved - Mathematics (M), Student (S), Tutor (T) and Author (A) – and the four faces – MST, STA, MSA and MTA – allow us to schematise and analyse the relationship between elements. In order not to excessively complicate the theoretical framework, we will not use the Tetrahedron in our analysis. Nevertheless, Albano (2017) highlights a crucial aspect: the presence of digital technologies induces an explicit centrality of the student in the learning process since it enables a dynamicity in the role that the student can adopt:

> «This means, in our systemic view, that the vertices of the tetrahedron are not static figures, but we consider them just as position, that is as a system's element, that can be played also by Student in some situations along the learning process. [...] Differing from the classical teaching, the didactic engineering should take into account the chance for students to move in other vertex-positions of the tetrahedron. This adds a dimension of dynamicity where learning does not come out from the fruition of a ready-made product but it is the outcome of a construction where students have chance to carry out didactic functions, suitably supported, guided and supervised, impacting on the didactic organization itself».
>
> (Albano, 2017, pp. 353-354).

In this study, we will refer to the Chevallard Triangle (hereafter: Triangle) whose general triadic structure will allow us to represent the different interactions and relationships between the protagonists of the two examples examined in sec. 4. In these examples, we will consider the aforesaid dynamism that allows the pupil (in particular) to occupy different positions within the model. Indeed, the Triangle is configured as a triadic relationship whose variables are the vertices of the triangle, which from time to time see the adoption of one role by specific individuals but in some cases can also see various individuals taking on different/multiple roles.

3.2 Classroom as society and interaction between physical and virtual T/L environments

We will follow the sociological interpretation of the classroom dynamics proposed by D'Amore (2005) and Bagni and D'Amore (2005). As a first step, we will focus on the meaning of the key term society, making reference, as the Authors do, to the

sociological definition. A society exists when individuals occupy a common area, interact with each other, know they belong to the same group, and have – or are supposed to have – at least part of the culture in common. In each society, two types of practices are generated: those substantiated by the fundamental purposes of that society, and those that derive from adaptation in order to belong to the society itself. According to the Authors, it is possible to think of the class – a group of people occupying the same physical learning space such as, for example, the classroom or the school – as a specific society of individuals «whose social unity is due to the necessity sanctioned by law for the implementation of defined and largely shared "practices"» (D'Amore, 2005, p. 5, translation by the authors). This society assumes meaning in the school context and therefore constitutes a social institution. In this society, some individuals take on the role of students and others take the role of teachers who, in this context, represent the institution itself.
Framing a class as a society (hereafter: class-society) allows one to refine the vision of the practices that take place in it. Thanks to the sociological lens proposed by D'Amore (2005), it is possible to distinguish between *functional* practices i.e. those that define a priori the class-society, and *extra-functional meta-practices*, which are more contextual and linked to a desire to belong to said society. The Author highlights that a class-society has T/L as its functional purpose, nevertheless, it does not mean that all the individuals – who are part of that class-society – share the same objectives or perspective. Thus, two types of groups can be identified: one of individuals with purposes, perspectives and *practices functional* to the specific society they belong to, and another of individuals who favour *extra-functional meta-practices* such as those generated by the didactic contract dynamics, defined by Brousseau (1986) as:

> «constituted [...] by the "specific habits of the teacher expected by the student" and by the "student's behaviours expected by the teacher" and precisely the choice of behaviour to be followed by the pupil becomes problematic since it leads the individual to question "what the teacher expects me to do"»
> (Brousseau, 1986, p. 66, translation by the authors).

In considering a T/L environment where digital technologies enable the interaction between physical and virtual learning spaces, further reflection is necessary. These conditions change the context by impacting on structural elements such as the definition of society itself and the ideas of class, classroom, practice and meta-practice. It is still possible to identify a common territory occupied by the individuals who populate this new learning space, but it is a multi-component and hybrid space due to the simultaneous presence of a physical component and a virtual component.
Furthermore, there are still some interactions between individuals among themselves and individuals with the environment, as well as the methods of carrying out the practices in which these individuals are involved. Nonetheless, these interactions take place both on a physical plane – which is strictly dependent on perceptual and spatio-temporal constraints – and on a virtual level – where the individual can control and manage not only the spatio-temporal aspects but also many-to-one communications. Indeed, virtual environments are characterised by a sort of evanescence of space-time boundaries in which participating individuals make decisions and act through their accounts. According to an infrastructural point of view, virtual environments force (allow) the individuals who are involved to make explicit that which

in physical environments could remain subject to unawareness or difficult to achieve. For example, in an T/L environment where physical spaces are intertwined with virtual spaces, the virtual learning space:

– can be used to connect people belonging to separate physical spaces (for example different physical classes, different schools), in order to create new societies, which are specific and distinct from the original society of the individuals involved, and where each society has its own practices and meta-practices;
– for example, it can allow: explicitly separating the teacher as a person from the actions that the teacher figure performs in a T/L context (such as designing activities and situations, managing them, evaluating etc.); it is possible to isolate each of these actions, and specific environments and/or accounts can be prepared for each; it is possible to decentralise the execution of these actions by distributing them among different individuals; etc. In this sense, the vertices Author and Tutor proposed in the Tetrahedron model are of great interest and they evoke a specification of possible actions carried out by the actors involved in a didactic context. These splits and redistributions of tasks and responsibilities force us to rethink (also as a result of any changes) not only the roles of the individuals belonging to the class-society (teachers and students) but also their relationships. Consequently, the changes will condition both how the individual may belong to the society (practices) and their attempts to adapt (meta-practices), with a possible impact also on pre-existing dynamics in the class-societies located in physical classrooms.

Finally, a virtual T/L space is substantiated in the dynamism and flexibility of the Triangle which allows the creation of class-societies following contextual and modifiable grouping criteria (Del Zozzo, 2019). These characteristics inevitably impact on (and problematize) the idea of class, of the classroom as well as the idea of practices and meta-practices.

3.3 Research questions

This paper aims to identify the contamination between physical and virtual classrooms. We set out to analyse situations where a T/L environment is contaminated by the presence of digital technologies for communication that enable the creation of a new T/L environment.

We highlight how the implementation of digital technology for communication in a physical environment which is already existent cannot immediately produce deep and effective intertwining between the physical and virtual classrooms; instead, it should be viewed as mutual contamination between the two classrooms.

To better understand the nature and the effects of this contamination on the T/L setting in mathematics, we need to answer the following questions:

Q1: How can the system which emerges from contamination between physical and virtual classrooms be characterised?

Q2: What can be identified as distinctive characteristics in order to track the evolution of T/L processes in the interaction between physical and virtual classroom?

In the next section we describe and analyse two implementation examples, while in sec. 4.3 we answer these two research questions.

4 Two examples of implementation

In this section, we describe two examples of T/L dynamics created in T/L environments where physical and virtual spaces interact with one another. Both episodes regard examples of communication *through* technology and they are not drawn from specific experimentations with their own design and methodology, but from real and everyday situations. The first episode involves an entire 10th year class in a upper secondary school of Applied Sciences, while the second regards two students in the last year of a upper secondary school of Classical Studies. For each of the two examples, we provide contextualisation and specify methodological and instrumental details regarding the interaction between physical and virtual contexts. We also describe the individuals involved, specifying their roles, actions and behaviours; we conclude by framing all these elements through the lens of our theoretical framework.
In both cases, we provide a posterior analysis and set out to relate the physical environment with the virtual one. We consider the analysed data to be truly interesting because the researchers' role is limited only to their collection and analysis without becoming involved in the design of the activities and their use in the new physical-virtual environment. Data are clean and they highlight some spontaneous behaviour of teachers and students in a T/L context which is hybridised by the presence of digital technologies for communication. Therefore, the two episodes function on one hand as examples of daily practices and, on the other hand, to show the interpretative potentialities of our theoretical framework. Since we have not adopted a standard experimental methodology, the data we use are partial and our analysis is based on these.

4.1 Virtual Classroom in the Google Classroom

4.1.1 Description

School and class: 10th grade class in a upper secondary school of Applied Sciences.

Context: The episode took place during the s.y. 2017/2018, when the school hosted a trainee who graduated in mathematics and specialised in the use of technology for educational purposes. The trainee conducted a coaching path for teachers and students about analysis, design and implementation of digital technologies in T/L. The school was registered on the G Suite for Education[3] and the trainee took the role of Administrator. For a *few weeks*[4], some teachers started to use Google Classroom[5] (GC) as an environment to manage virtual classrooms. The example involves a single class which started to use GC with the mathematics teacher to manage a learning process about radicals which had been set up during the previous weeks. At the end of this path, the teacher, following a suggestion from the trainee, proposed to the students that they use the GC Stream[6] as a shared space where they could share

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4 – 5
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3. https://support.google.com/a/answer/2856827?hl=it.
4. With respect to the episode we report here.
5. https://support.google.com/edu/classroom/?hl=it#topic=6020277.
6. It is a communication space visible to all members of that class in GC, whether they are teachers or students. To learn more: students point of view at the link https://support.google.com/edu/classroom/answer/6020274?hl=it&ref_topic=9049835; teachers point of view at the link https://support.google.com/edu/classroom/answer/6020270.

their doubts, difficulties and questions. The curricular teacher involved in this example had been teaching this class since their first year and the only digital technology she used prior to the introduction of GC was the electronic register in order to send homework to students and to report test marks. Therefore, the introduction of GC was integrated into a pre-existent didactical context whose characteristics (about relations, dynamics of didactical contract, habits etc.) were already delineated solely according to the physical classroom environment.

Digital technology involved: The software component is GC with all its features. The hardware component can be PC, tablet or smartphone depending on the device used by the individuals interacting in the platform.

Physical people and accounts involved: 26 physical people: 22 students, 1 mathematics curricular teacher, 1 trainee, 2 teachers of other curricular subjects (passive). 27 accounts, one for each individual, in addition to a passive account managed by the trainee and added in GC as a "fake student" in order to monitor the entire flow of activities in the platform. Net of passive entities, we consider only 24 physical people and their respective accounts. Hereafter, S_i represents a person-student of the class in question.

Interaction between physical and virtual levels: The correspondence between individuals in the physical classroom and in the virtual classroom is almost one-to-one. The only exception is the trainee who belonged generally only to the second context and who, in the physical classroom was present only very occasionally. The trainee had neither a voice nor role in evaluation. The accounts in GC were denominated by complete name and surname for all those involved; thus an explicit identity correspondence existed between individuals in the physical environment and accounts in the virtual setting.

Physical classroom	Virtual classroom
1 curricular teacher	2 active teachers (curricular and trainee)
22 students	22 students
1 trainee (she had a transversal role and was present in the physical context only sporadically)	

Table 1
Individuals who populated the physical classroom and the virtual classroom.

When: the episode took place between 24th and 28th November 2017.

What happened:
In order to simplify the references in **sec. 4.1.2**, the passages are enumerated.

1. In the physical classroom, the teacher showed pupils how to use GC.
2. On the 24th November, the teacher assigned some exercises on the electronic register, assigning at the same time the following task in GC: «Post a photo of all the exercises in which you had some difficulties and, in the event you did not encounter any problems, post at least one completed exercise».

Between 26th and 27th November:

3. S_1 accepted the teacher's invitation regarding use of GC in order to help each other in doing homework and he posted a photo of the book page on the Stream, highlighting two exercises where he encountered difficulty;
4. S_2 posted a photo of a page of his notebook on the Stream, with the exercises that he had attempted accompanied by the following message: «Photo of exercises for 28th (I was not able to complete any of them)».
5. Three more students followed S_2's example and posted on the Stream a photo of the page of their notebook with their exercises solved.
6. Another student, S_6, followed S_1's behaviour and posted a photo on the Stream of the page of the book without sharing his attempt at the exercise.
7. S_7 posted the following text message to the Stream, accompanied by the photo in **Figure 3**: «I was able to do the exercises – if you want you can see them; I wrote in pencil the passages that I skipped, so you can understand them».

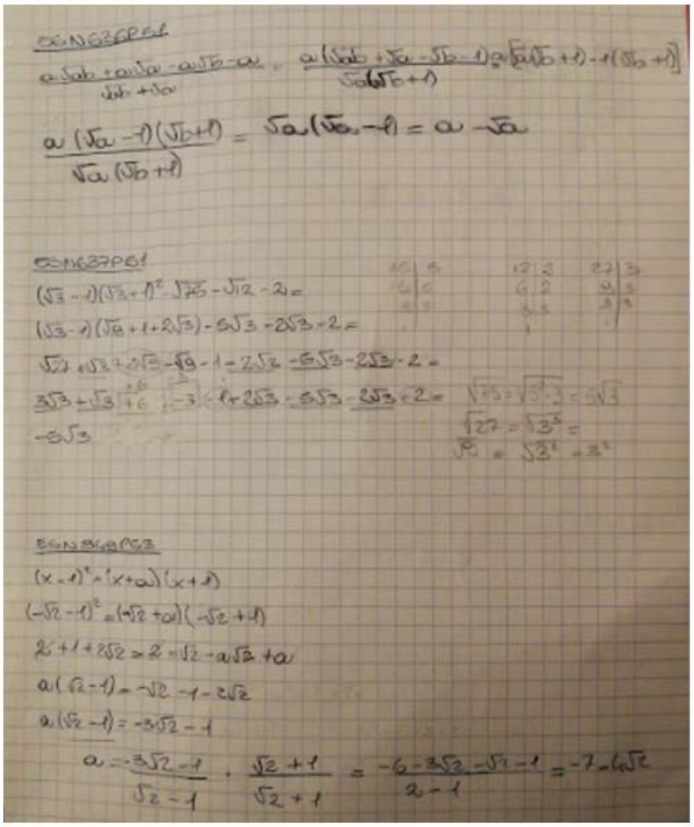

Figure 3
Photo of S_7's notebook that she attached to her post in the Stream.

On 28th November:

8. The teacher thanked the students who shared their notebook, and she explained the usefulness of this action in highlighting difficulties and encouraging mutual help. The teacher answered S_7, thanking her and hoping that her detailed exercises could be of help for her classmates. In the Stream, there are no other answers to S_7's Post.[7]

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7. We highlight that GC, at least up to the moment of writing, does not notify other students when one student publishes a post, unlike when the post is written by the teacher. (https://support.google.com/edu/classroom/answer/6141557?co=GENIE.Platform%3DAndroid&hl=it). This means that probably none of the classmates noticed S7's Post in good time.

4.1.2 Interpretation

Chevallard's Triangle

The interaction between the physical and the virtual level enables the fulfilment of two configurations that can be highlighted or, rather, that can be revealed by using the Triangle model.

In fact, as we outline in **Figure 4**, in this case, we can distinguish two instances of the Triangle: one is for interpreting the episode in the context of the physical classroom (T_{CF}) while the other frames the episode in the context of the virtual classroom in GC (T_{GC}).

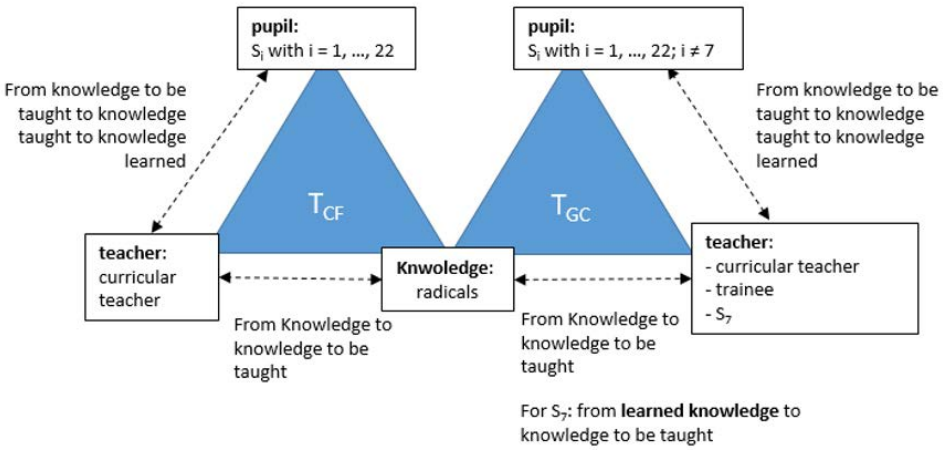

Figure 4
The schematisation of example 1 both in the physical classroom context and in the GC context.

If we only consider the curricular teacher, thus referring to the Triangle on the left in **Figure 4**, we can see a rather standard didactic sequence. However, if we consider the Triangle on the right, the GC presence enables a dynamicity which provides more complexity. This dynamism allows students to move between the teacher and the pupil Triangle vertices. S_7 accepts the proposal that the teacher made in the physical classroom. Then, by responding to the proposal in the GC, S_7 takes responsibility for her own learning and shows her didactic intention towards her classmates. In T_{GC}, S_7 assumes a teacher role and achieves a Didactic Transposition from her personal learned knowledge to a knowledge to be taught to her classmates (line 7: « I was able to do the exercises – if you want you can see them; I wrote in pencil the passages that I skipped, so you can understand them»).

Triangle instances such as the T_{GC} have peculiar characteristics which should be studied more in depth. Indeed, we cannot locate a didactic contract with its clauses and effects in the classical sense (D'Amore, 1999) since in peer education - which characterizes this new t-p-K configuration – the system of rules, behaviours, expectations, interpretations and beliefs has a different nature from that of the same system in the physical classroom with the curricular teacher. Moreover, S_7 is a student in T_{CF} and her learning is reactivated by a process of personal re-construction of knowledge in a communicative lens in order to help her classmates who, in the GC, are also her students. Thus, with her post in **Figure 3**, S_7 fields the typical processes that regard the relation between teacher and student both in the T_{GC}, where she assumes the teacher position (Didactic transposition) and in the T_{CF}, where S_7 assumes the student position (devolution, personal implication, knowledge construction and a break in the didactical contract).

Sociological interpretation

Using a sociological perspective, two class-societies appear to emerge, which are distinguishable but not separable.

The first class-society is the physical-class-society in which the individuals involved are the 22 students with their teacher; the common territory occupied is physically demarcated and the interactions are scheduled a priori; the practices and meta-practices have been delineated before the introduction of GC.

In the physical-class-society, the fundamental practice is the T/L of mathematics and in this society some meta-practices are developed, which involved both students and the teacher. For instance:

– Students attempt to satisfy their need for socialisation, the need for reinforcement and recognition of their identities, and need for approval.
– The teacher is concerned with satisfying the demands of the noosphere, about completing bureaucratic tasks, the execution of the program, the gathering of evaluations; not only the students but also the teacher acts to achieve affective and relational objectives as, for example, when s/he exerts control over the students, when s/he wants to satisfy the demands of the school board/management, when s/he is worried about safety and security, or when s/he transfers into the classroom all her/his beliefs concerning mathematics, students, school and society. All these aspects interfere and/or can conflict with the fundamental practices of this society.

The second class-society is the GC-society composed of 22 students, their teacher and the trainee; the common territory is digitally demarcated by registration in the GC that the teacher has created; the interactions occur in a personalized time and manner; the dynamism – of the context and interactions – implies a dynamism in practices and the meta-practices which are thus flexible as opposed to constant. Furthermore, even though there can be different functional practices which define the class-society, the fundamental purpose of this second society is the same as that of the physical-class-society: T/L in mathematics. Regarding the practices and meta-practices in the physical-class-society, it should be borne in mind that in the context in question, this society evolves after the physical-class-society where, *before G*C, students' performance was examined by the teacher only during tests and evaluations. The episode presents four distinguishable behaviours that we interpret in terms of practices and meta-practices; these behaviours highlight various relations between physical-class-society and GC-society:

1. S_1 e S_6 posted the book page and pointed out the troublesome exercises for them; in doing so, they *located their difficulties without exposing themselves* by showing their attempt at a solution in the Stream.
2. The four students from S_2 to S_5 posted a photo of their notebook with their failed attempt; by doing so, they *located their difficulties and showed their logical process to all the individuals in the GC-society*.
3. In order to answer her classmates, S_7 posted a photo of her notebook with her attempts which she declares correct. Moreover, S_7 expresses her didactic intention: she is doing more than simply presenting her work *to all the individuals of the GC-society* since *she is offering her work as an example for her peers*.
4. The remaining 15 students abstained from participating in GC, each with their own motives (no difficulties, not willing to expose themselves, some technical or connection problems etc).

These behaviours highlight strong relations between practices and meta-practices in the physical-class-society and the GC-society. The way S_1 and S_6 act can be seen as a meta-practice in the GC-society which – extensionally – is inherited by a meta-practice in the physical-class-society that arises from fear of evaluation and the dynamics of the didactic contract.

The way students belonging to the 2nd category act is, on one hand, comparable to a fundamental practice in GC-society, while on the other hand, it can be seen as a meta-practice that arises from a desire to satisfy the teacher's request regarding the sharing of students' doubts and difficulties. Indeed, these students' behaviour reveal a partial breach of the didactic contract; students do not show personal implication or an intention to build their personal knowledge – which would be seen if they had asked specific questions to trigger a discussion or if they had shared their solution strategies.

As we previously mentioned, the spontaneous action of S_7 highlights personal implication and responsibility which can be considered as a typical practice in a GC–society, coherent with the fundamental purpose of this society (T/L of mathematics).

Nonetheless, the Stream can be viewed by the teacher as well; thus, we can interpret the behaviour of S_7 as a meta-practice regarding the physical-class-society. Indeed, S_7 displayed her work – while declaring its correctness – not only to her classmates but also to the evaluator-teacher.

Regarding the 4th category, in our practices and meta-practices analysis, it cannot be ignored that 15 students do not interact in GC. In order to configure this element in terms of practices and meta-practices, we should have more data; nevertheless, a group of 15 individuals in the GC-society is still qualitatively and numerically relevant.

The dialogue between the physical-class-society and the GC–society triggers a new hybrid T/L space: the individuals here are the 22 students, the teacher and the trainee; each person is aware of occupying a shared space whose boundaries shift continuously due to the contamination of the first two environments. On one hand, the interaction between individuals and the communicative flows among them are better controlled and arise from explicit intentional processes; however, on the other hand, the same processes have increased spontaneity since each person has the opportunity and adequate space/time to express (or NOT to express) him/herself.

Furthermore, the discretization typical of digital environments enables the recognition and reconstruction of each participant's actions, which allows the individuals involved to make focused and conscious decisions.

Everything that occurs in the hybrid T/L environment has an unavoidable relationship with the other two societies; it is in contact with the practices and meta-practices that are already present, highlighting them, sometimes modifying them, or in some cases, crystallising them – see, for instance, episode 2 in Del Zozzo and Santi (2019).

Conclusively, the introduction of GC enables communicative scenarios and semiotic resources which are different from those that characterise the physical classroom; nonetheless, we wish to highlight, in this example, the poor quality of spontaneous interaction among students in the GC. Indeed, the interaction is limited almost exclusively to uploading materials in GC without any exchange between students. Under the theoretical framework we chose, this is interesting data. We have to consider that the students have only recently started to use GC and they do not yet have enough experience to interact efficiently with this technology. In the situation described, students' behaviour indicates practices and meta-practices they inherit from the physical classroom, such as those caused by the didactic contract.

However, the choice of material to be shared is not as neutral as one might think. Indeed, in contrast, we described earlier some behaviours which can be interpreted in terms of practices and meta-practices, where we reaffirm our interest for the actions of S_7, which dynamize the triadic relation of the Triangle as S_7 shifts from a student role in the physical-class-society to the teacher role in GC-society.
The short period we have considered here does not provide us with enough information about the T/L dynamics in the emergent space in order to delineate them precisely. Nevertheless, practices and meta-practices of the two society are not isomorphic; on the contrary, they contaminate one another and so we believe that the study of their relationship should be a further line of research.

4.2 Private WhatsApp chat between two students

In the next section, we describe and analyse an exchange of messages on a mathematical topic between two classmates, S_1 and S_2,which took place in their private WhatsApp chat.
We were able to access these data since one of the authors of this paper knows personally, in a private dimension, one of the two protagonists. The author asked for explicit permission to access these data. The two students consulted each other about the request and, with enormous generosity, they welcomed our invitation, authorising us to transcribe the parts of their chat where they were talking about mathematics.
The exchange we report here dated back to one year and a half before we asked them for permission for use and it was possible for us to access it as one of the two students did not delete the chat memory in the phone. When the two students had the actual conversation that we describe here, they did not know nor could they have imagined that somebody else would have read/seen/listened to their conversation. Thus, the following is an a posteriori analysis of a private communicative exchange which genuinely occurred between S_1 and S_2.

4.2.1 Description

School and class: last year of a upper secondary school of Classical Studies.

Context description: The class was preparing the last mathematics test before the final State Exam and the teacher assigned the students some revision exercises. Some of the students in the class flanked their work at school with group afternoon lessons with the same private tutor (who is not their curricular teacher). The episode we consider here is based on a WhatsApp chat between two students, S_1 and S_2, who shared both the curricular teacher (as classmates) and the private tutor.

Digital technology involved: The software component is WhatsApp with all its features. The hardware component is the smartphone.

Physical people and accounts involved: 4 physical people: the two students S_1 and S_2, their private teacher (who is present implicitly only) and another adult (also present implicitly) who is a family member/friend and a mathematical point of reference for S_2; 2 WhatsApp accounts (belonging to S_1 and S_2).

Interaction between physical and virtual levels: S_1 and S_2 agreed to do some exercises by Skype together. On the same day, S_1 followed a lesson with the private teacher that S_2 did not attend. The following day, S_1 and S_2 contacted each other via WhatsApp for an update about the private lesson.

Physical classroom (at school)	Private physical classroom	Familiar private context	Whatsapp
1 curricular teacher 19 students	1 private tutor 4 classmates (they are not always present all together)	1 family member/ friend who is a mathematical point of reference 1 student	2 students

Table 2
Individuals who populate the three physical environments (the physical classroom at school, the private physical classroom, the familiar context) and those who communicate via WhatsApp chat.

When: the episode took place between 23rd and 24th May 2018.

What happened:
In order to simplify the references in **sec. 4.2.2**, the passages are enumerated.

– S_1 started the conversation in order to inform S_2 that he had done all the exercises during the private lesson; S_2 told S_1 that she was working on the same exercises with an adult family member/friend – who is for her a point of reference in mathematics – and then she wrote to S_1: "Could you send me the explanations??"
– S_1 sent S_2 five photos of his notebook with all the work done during the private lesson. Among the exercises, there was also the following multiple-choice test (n. 4 of page 1484):

> The function $f(x)=-x^5+2x^3$
> A. Has one stationary point of inflection
> B. Has one stationary and two non-stationary points of inflection
> C. Has three stationary points of inflection
> D. Has three non-stationary points of inflection
> E. Has no point of inflection

S_1 solved the exercise as we report in **Figure 5**:

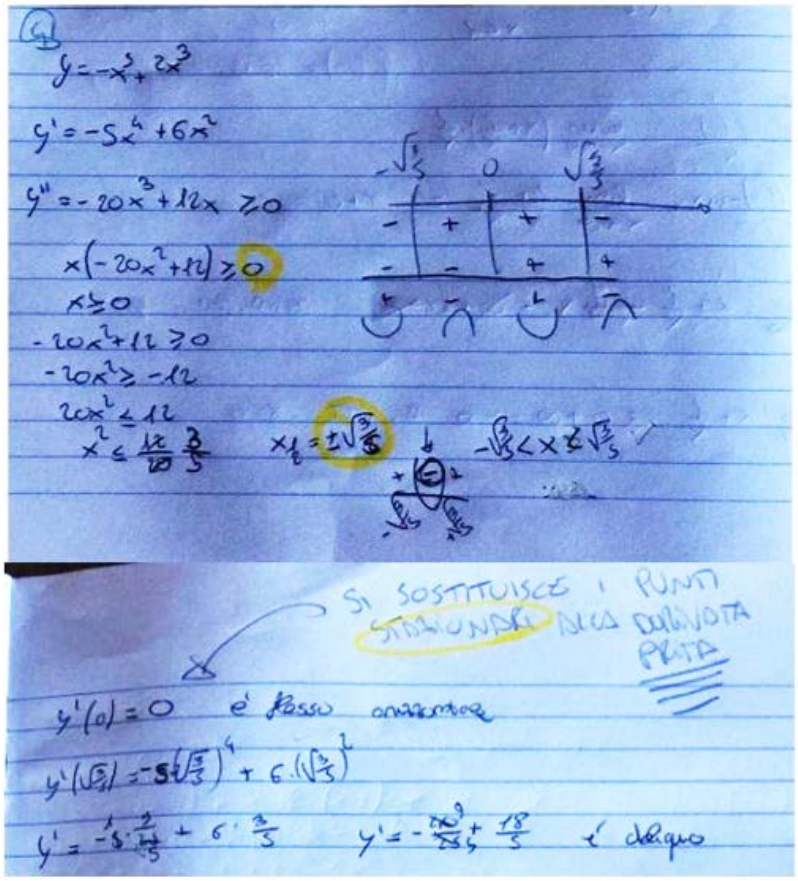

Figure 5
An excerpt of the photos that S_1 sent. This Figure shows S_1's solution to the multiple-choice test on the point of inflection.

3. After some other messages unconnected with mathematics, S_2 sent the audio message that we transcribe here:

> «The exercise on page 1484, number 4, when you have to solve the function and you have to find the maxima, the minima and the points of inflection and so on, at a certain point you wrote the table, the one with the signs, and you put in there only... I mean I... how do I work out... mmm... I mean which ones are the mi... I mean, in the sense... tell me that you understand».

This audio message is accompanied by an extract of S_1's notebook where S_2 circled a section:

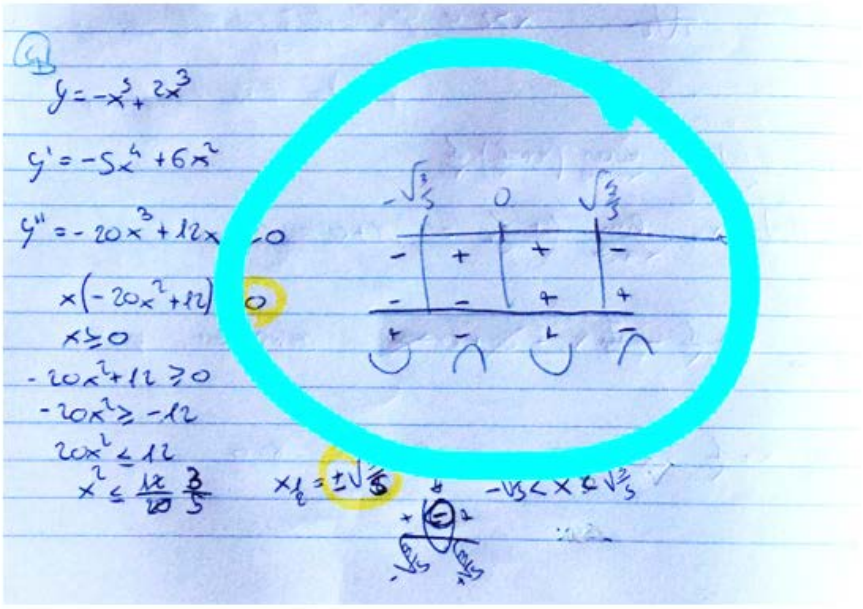

Figure 6
Image sent by S_2 which accompanied the audio transcribed in line 3.

4. S_1 answered with the following audio message:

> «mmm no, I don't understand but, ok, I'll try to explain it to you; basically, you find maxima and minima with the first derivative. You find the concavity with the second derivative. Since n. 4 looks for the points of inflection, I mean the points of inflection, the concavities, you find them with the second derivative, so now you have to do firstly the first derivative and then the second derivative, and on the second derivative you study the signs. Then when it is greater [*than 0*], the concavity is upwards, when it is lower [*than 0*] the concavity is downwards, so since the concavity is changing, there is a point of inflection, ok? Oooh. Then, when you have done this, to understand if it is a stationary or non-stationary point of inflection, you have to do what I wrote for you in pencil in the photo, which is to substitute the stationary points, which are 0 plus or minus square root of three-fifths, to the first derivative [*emphasis in the voice*], if it is 0 it is a stationary point of inflection, do you understand? In fact, by substituting 0 in the first derivative it is equal to 0 and it is a stationary point. If I substitute plus or minus square root of three-fifths, it is... it is not 0 and so it is... non-stationary. It's like that both for plus and for minus. So [*emphasis in the voice*] there is one stationary and two non-stationary points of inflection».

5. S_2 wrote «Ok, perfect» and then, after 5 minutes, she asked: «What about n. 6?»[8]
6. S_1 sent the following audio message:

> «Since you must find which one of them, the functions under there... eh... regarding those points... eh... it regards those points and you for example fin... a... tr... g [*incomprehensible mumbling*] the first point is the domain plus or

8. The exercise S_2 is referring to, is a multiple choice question in which it is necessary to decide which of the five functions listed is the one that satisfies a series of pre-established conditions.

> minus 4 which means x is different from plus or minus 4 [*questioning tone*] so you go to check which of the five functions has the domain different from plus or minus 4. So, for example it cannot be A, it cannot be E, nor C. So already with these... ehm... you can exclude at least 3 based purely on the domain. So the question remains between B and D. Ehm "It intersects the x axis in points A and B" [*it seems as though he is reading*] so you have to intersect B and D with the y axis, for example, and if you do that you see the two intersect the points A and B. Which means that both intersect there. So you have to check the point 3. "It has as vertical asymptotes the lines x=4 and x=-4 " [*appears to be reading*] and you make the limit for x the limit for x as x approaches 4 and... ehm... the limit for x as x approaches –4 and also both B and D so you have to check the point 4. "It has as horizontal asymptote the line y=2" [*it seems as though he is reading*] so you do the limit for x the limit for x as x approaches infinite of for example B and then you see that it is infinite over infinite ad so... ehm... it is not that and D has the horizontal asymptotes. And so D is the answer. Do you understand? I mean, you have to eliminate possibilities, you start from points 1, 2, 3 and 4 to the functions and and you exclude them as you go along».

7. S_2 wrote «Ok, thanks» and the conversation ended.

4.2.2 Interpretation

Chevallard Triangle

In this example, teacher, pupil and Knowledge interact in the 4 different configurations we present in **Figure 7**. The first Triangle interprets the physical classroom context (T_{CF} on the top-left of the image); moreover, the analysis also calls on:

– A second Triangle to interpret the triadic relationship between the private tutor, Knowledge (analysis of functions) and the pupil S_1 (T_{IP} on the top-right of the image);
– A third Triangle to interpret the triadic relationship between the teacher – familiar adult and mathematical point of reference for S_2, Knowledge (analysis of functions) and the pupil S_2 (T_{RC} on the bottom-right of the image);
– A fourth Triangle to interpret the triadic relationship between Knowledge (analysis of functions), the teacher S_1 and the pupil S_2 (T_{WH} on the bottom-left of the image); this situation recalls that of the previous example with S_7 albeit with a crucial difference: there is no curricular teacher involved but S_1 behaves as though he were the teacher, S_2 behaves as the student, and there is a Knowledge content at stake.

Looking at **Figure 7** once more, we highlight the different knowledge starting points of the individual playing the teacher role in each context. For instance, in T_{RC} the teacher-familiar adult is just a few years older than S_2 and he has already finished upper secondary school. Therefore, even though his personal knowledge may be approaching academic level, it cannot be considered equivalent.

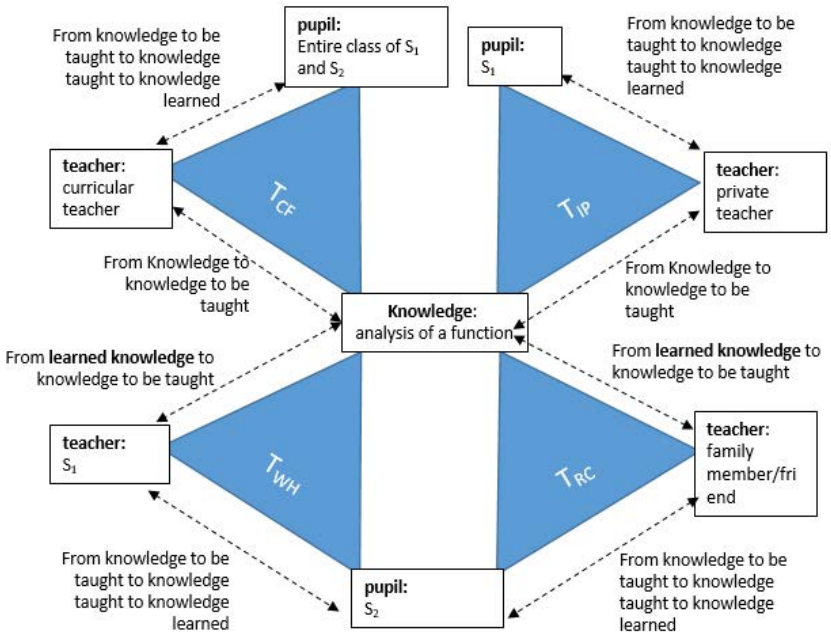

Figure 7
Interpretation of example 2 in terms of Triangle modelling.

In this example, the Triangles are not so enmeshed as in the example in **sec. 4.1**: in the four Triangles, the teacher role is adopted by 4 different people, one of whom is a student; moreover, the curricular teacher here is present only in the physical classroom as opposed to the previous example where the curricular-teacher was present also in the GC; finally, in this example, only two students of the institutional physical classroom (S_1 and S_2) are present in all the Chevallard triangulations. In this section, we limit our analysis to the learning context which arises from the relationship between S_1 and S_2 in their WhatsApp chat. This context lacks the curricular teacher, yet the situation is still of high didactic relevance. The conversation we report regards a discussion between two students upon two exercises, and we can organise the content in two parts: from line 1 to line 4 (inclusive) the conversation regards the first exercise; from line 5 to line 7 (inclusive) S_1 and S_2 talk about the second exercise. Throughout the conversation, the S_1 figure appears emblematic: he communicates his intention to help S_2 in solving some exercises and he performs a devolution of didactic objectives that S_2 accepts, involving her in the mathematical activity. As can be seen in the previous episode with S_7, here we also witness a re-construction of the knowledge previously learned by S_1, which he triggers in order to help S_2 learn. As in the first example with S_7's behaviour, we can see how S_1 modifies his work in order to maximize its communicative potential (line 4: «[...] you have to do what I wrote for you in pencil in the photo [...]»). Furthermore, in triangle T_{WH}, where S_1 assumes the teacher role, his actions can be interpreted as a kind of Didactic Transposition, although not in the traditional form that starts with Academic Knowledge and ends with the student's competent knowledge - here the sequence begins with the knowledge learned by S_1 in the combination of T_{CF} e T_{IP}. The S_1 didactic goal impacts on S_2 who manifests a critical attitude and a learning intention. In the discussion about the first exercise, S_2 exposes her difficulties in a language – not part of the typical contractual jargon that D'Amore (1999) calls "matematichese" – that is apparently incomprehensible to anyone not involved/informed of the communication between S_1 and S_2 yet clear to S_1. We would highlight the active role of S_2, who

even takes on the responsibility of modifying S_1's materials, which is an improbable operation if we think about the contractual dynamics typical of education systems where the presence of a teacher can activate all the expectations and effects studied in the literature.

We also draw attention to the role of digital technologies that are essential in order to achieve the didactic configuration under consideration here. Indeed, digital technologies offer the opportunity to take a photo of the notebook, to modify the digital content semiotically, to send these documents via internet; the above operations may be considered obvious but we invite the reader to imagine how the situation could have evolved without such technology. Moreover, from a research point of view, the nature of these kinds of technologies allows us access to a spontaneous communicative exchange between two students, which otherwise would be accessible only at the cost of unavoidable interference by the experimental contract (D'Amore, 1999).

Regarding the discussion of the students about the second exercise, we do not see the same elements of personal implication and learning by S_2. Indeed, S_2 limits herself to pointing to an exercise without any expression of will or intention to learn a specific item of knowledge. This behaviour reminds us of the behaviour of S_1 and S_6 in the first example from **sec. 4.1** of this paper. Contrary to what can be observed in the first part of the two students' communication, some didactic obligation has probably intervened in the second part of the conversation – for instance, the delivery of the exercises, the exercise as a topic for oral examination, anxiety about the upcoming test etc.

In S_2's behaviour, we see a shift towards attitudes guided by didactical contract dynamics, almost as though a transfer of the didactic contract from the physical classroom is unavoidable. However, with the data currently at our disposal, we cannot proceed further in this analysis.

Sociological interpretation

According to our analysis, in this episode, we can distinguish two class-societies and we can describe the interaction between S_1 and S_2 in terms of practices and meta-practices. On one hand, we have the physical-class-society of the school institution, which is populated by the curricular teacher and all her students. The common territory occupied is physically demarcated and the interactions are scheduled a priori. On the other hand, we have the physical-class-society in the non-institutional context which is populated by the private tutor and all her students. The common territory occupied is physically demarcated here also, and the interactions are scheduled from one time to the next among the members of this society.

Even though the two class-societies involve different people in different territories, they both share the same fundamental practice: the T/L of mathematics. In both class-societies, practices and meta-practices have been delineated over time. In this scenario, S_1 and S_2 are two individuals who belong to both class-societies and they have a friendship between equals that, in itself, is completely centred on the interests of the involved individuals.

In the example we are analysing here, the 2 students communicate about the two exercises in an intimate digital space – which is not a class-society per se but is affected by practices and meta-practices of the two societies described earlier – a space that is dedicated to their friendship. In this private space, the conversation about the first exercise presents substantial differences from the conversation about

the second exercise. Indeed, the first part of the conversation arises from a specific doubt around which S_2 expresses a specific question referring to her own needs: (line 3: «how do I figure out...[...]»). On the contrary, the conversation about the second exercise is triggered again by S_2, but the question, when compared with the previous one, has a completely different tone (line 5: « What about number 6?») as if it contains also an impersonal and tacit "how should it be done".
Naturally, considering the friendship between S_1 and S_2, there is no difference between the two parts of the conversation since both represent an exchange of assistance which fits perfectly within such a type of relationship.
Nevertheless, in terms of practices and meta-practices concerning the physical-class-society to which both students belong, only in the first part of the conversation does S_2 refer to herself personally and manifest learning intentions; in this first part, we witness an assonance with the functional practices of the class-society while the second section reveals a sort of dissonance with these practices. Indeed, in the second part S_2 loses her active role and there is a shift towards attitudes deriving from didactic contract dynamics.
As in the first example, here also the two class-societies have a non-empty intersection.

4.3 Answer to the research questions

Q1: How can the system which emerges from contamination between physical and virtual classrooms be characterised?

A1: In order to define this new system, we have to identify the possible cases of the variables t-p-K which contribute to the the triadic relationship embodied by the Triangle. For instance, the episode in **sec. 4.1** presents two possible cases of the Triangle (**Figure 4**). In the episode in **sec. 4.2**, the complexity of relations which exist among the protagonists has been framed in four triadic schemas.
The identification and description of the class-society is a further element that has to be taken into account. More precisely, for each class-society it is necessary to recognise the individuals/accounts, the territories, and the "interaction rules" between the individuals/accounts. For instance, in the first episode we identify two class-societies:

– The *physical classroom* which is defined by the students and their teacher; all of them acting in a common territory demarcated in space and time according to the typical rules of a school context.
– The *GC* which includes 24 individuals/accounts: 22 students, the teacher and the trainee. The common territory has no defined spatial and temporal limits. These boundaries depend both on the choice of the individuals and on the characteristics of the digital technology in use. The rules of interaction are defined both by agreement between individuals and the constraints of hardware/software components of the technology.

In the second episode we identify two class-societies:

– On one hand, there is the *institutional physical classroom* at school, constituted by the curricular teacher and all the student members. The common territory occupied is physically demarcated and the interactions are scheduled a priori.
– On the other hand, there is the *non-institutional private physical classroom*, which is defined by the private tutor and a subset of the students from the class in question. Similar to the other society, here too the common territory occupied

is physically demarcated, but here the interactions are scheduled from one time to the next by the members of this society.

We highlight that the two class-societies, despite involving a different set of individuals in different territories, share the same fundamental purpose: the T/L of mathematics.

Q2: What can be identified as a distinctive characteristic in order to track the evolution of T/L processes in the interaction between the physical and virtual classrooms?

A2: In order to answer this second research question, we must identify some dynamic factors which enable us to follow the evolution of the T/L process. These factors are tied to the theoretical lens we are using to interpret the reciprocal contamination of physical and virtual classrooms.

Firstly, as regards the Triangle, it is important to consider the dynamicity of the triadic relationship concerning the variables t-p-K. As we highlight in **sec. 3**, the introduction of virtual classrooms enables a dynamicity in possible configurations of the Triangle with respect to the variables t-p-K. The teacher vertex is not an exclusive pertinence of the curricular teacher, but it represents a role that can be adopted by other individuals/accounts as well. There are configurations of the Triangle in which not all the students of the same class belong to the pupil vertex since they can distribute themselves between the pupil vertex and the teacher vertex; moreover, the teachers themselves can assume a student role as regards, for instance, the usage of digital technologies. The knowledge vertex is not occupied by Academic Knowledge exclusively, but can include different forms of knowledge envisaged by the different phases of Didactic Transposition.

In the first episode we show that in the shift from physical classroom to GC, the Triangle evolves from the initial situation – where the vertices t-p-K are respectively occupied by the mathematics teacher, the students in the class and Academic knowledge – to another situation where this configuration has been shifted to the GC and where the teacher vertex is taken over by S_7 the student vertex is constituted by the remaining students of the class who are participating in the discussion, and the knowledge is the personal learned knowledge of S_7; it is from this knowledge that the Didactic Transposition arises.

In the second episode, **Figure 7** represents the complexity of evolution in the T/L processes. The variables t-p-K of the triadic relation are embodied in the Triangle dynamicity:

– by the curricular mathematics teacher, by the private mathematics tutor, by the competent adult family member/friend and by S_1 (as regards the teacher vertex);
– by the entire class, by S_1 and by S_2 (as regards the pupil vertex);
– by Academic Knowledge and learned knowledge (as regards the knowledge vertex from which the Didactic Transposition starts).

Secondarily, the type of practices that characterise the class-societies which emerge when physical classrooms and virtual classrooms contaminate one another is another dynamic factor that has to be considered in order to trace the evolution of T/L processes. In particular, we are referring to the functional practices, extra-functional meta-practices and their mutual relationship.

In the first episode, we identify dynamics which derive from the interaction between functional practices and extra-functional meta-practices – dynamics that evolve to satisfy the invitation of the teacher regarding the exercises to be posted in the GC Stream. The first exercises posted in the GC Stream show how meta-practices of

the physical-classroom society are interfering with the basic functional practices in a GC-society. Indeed, the first students highlighted only those exercises that they were not able to solve – which means that they exhibited shyness about posting their solution publicly to the teacher and classmates. One meta-practice – which is inherited from the physical-class-society and can be associated with the didactic contract – is interfering with a functional practice that consists in the learning of mathematics. Nevertheless, in the GC-society we notice an evolution from a meta-practice to a functional practice, triggered by S_7 who (in the context of the virtual classroom) takes on the teacher role and operates a shift from learned knowledge (by S_7 herself) to knowledge to be taught, to knowledge taught to learned knowledge. Probably, in the setting of the physical-class-society – where the triadic relation t-p-K is more static than in a GC-society – S_7 would have not found in herself or in her classmates the emotional and affective support necessary to break the didactic contract.
In the second episode, we observe a shift in S_2 behaviour from a functional practice towards an extra-functional meta-practice. Indeed, S_2 starts her interaction with S_1 with a functional practice aimed at the learning of mathematics – S_2 wants to comprehend the exercise about the analysis of a function. However, when she deals with the second exercise, her behaviour is attributable to an extra-functional meta-practice as outlined in sec. 4.2.2.

Finally, the possibility of disentangling the complexity that Borba et al. (2016) highlight, and which emerges when digital technologies encounter physical learning environments, has to be ascribed both to the epistemological strength of the Triangle and to the notion of practices and meta-practices that D'Amore introduced in Mathematics Education. According to our analysis, these two theoretical lenses allow us to embrace both the dynamicity and the systemic-relational complexity which emerge from the interaction of physical and virtual classrooms.

5 Conclusion

In this work, we analysed two examples of T/L dynamics in two environments enriched by the usage of digital technologies, and we proposed a theoretical framework made up of two elements:

– the triadic scheme of the Triangle, in order to frame the character and relations between protagonists, whose dynamicity is typical in T/L contexts which host some digital technologies (Albano, 2017);
– the sociological perspective of the classroom as a society introduced by D'Amore that we used to relate the practices and meta-practices in the emerging class-society.

The first episode we considered was an exchange of posts in the GC Stream by a 10th grade class of a upper secondary school of Applied Sciences. The second episode we considered was a extract from a private WhatsApp chat between two classmates in the last year of a upper secondary school of Classical Studies. Both examples regard the interaction between the environment of physical and virtual T/L environments and they are authentic episodes that we have analyzed a posteriori. In order to frame and study such type of situations even from a real context situation,

we propose in this work a theoretical approach and perspective. The triadic schema of the Triangle allows us to represent (from a didactic perspective) all the interactions between the individuals involved. The concept of the classroom as society allows us to approach a complex situation, such as that where physical environments interact with virtual ones, through an analytical lens. In our perspective, this lens permits us to define the common territories, the individuals/accounts who are involved, the interactions and behaviours (that we have seen in terms of practice and meta-practices), and it is an approach that helps provide a clearer didactical analysis.
The two examples, albeit very different from each other, revealed some common points through the theoretical analysis that we were able to identify and specify. The first point concerns the inevitable contamination between what occurs in the context of the physical classroom and what happens in virtual environments. This contamination emerged as stronger in the first example, in which the physical and virtual settings spawned a new emerging space that sees them almost intertwined, but is still present in the second episode, in which the physical classroom didactic contract manages to infiltrate even into the private space of two students. The second point concerns the presence of elements – such as devolution, personal implication, construction of knowledge and breaking of the didactic contract in the interactions – that occur in virtual spaces. Naturally, this does not mean that such elements exist only in the aforementioned spaces, nor that they always occur in such but – in our opinion – the extremely interesting point is that in the virtual spaces they are visible, they become observable and investigable. The third point, linked to the second and which unites both the examples reported, concerns the difficulty of clearly distinguishing practices from meta-practices, especially in the hybrid contexts in question. Indeed, the fluidity and dynamism that characterise these contexts not only increase the complexity of the system under analysis but they also blur its boundaries.
With respect to the aforementioned work of Borba et al. (2016) – which highlights how digital technology has completely deconstructed the image of traditional learning environments, enabling new types of classes and ways of working which, in turn, trigger different socio-cultural dynamics – the two episodes that we have analysed show how digital technologies have been decisive in modifying the nature of the Triangle and the Didactic Transposition. Nevertheless, we believe that research in this area should continue by strengthening the role of digital technologies in modifying another fundamental dimension that distinguishes the learning of mathematics: we refer to the very rich world of semiotics in its structural-functional and semiotic-cultural variations.
Finally, we would like to end this work not so much with answers but rather with some general reflections and the highlighting of a need to better define research problems in this area.
In a T/L context, digital technologies act similarly to a prism that refracts light: they break down subjects, objects and teaching processes, unpacking and making their various subcomponents visible (and, thus, possible to analyse). In this sense, the second example we consider is emblematic: the collection and storage of data are implicit when using tools such as WhatsApp and, thanks to the invaluable contribution of S_1 and S_2, we had the opportunity to study an authentic private conversation between students without having it contaminated by our presence, whether declared or hidden. We believe that these characteristics have, and will increasingly have, a significant impact in terms of research methodology in mathematics education.

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