On students' *concept-image* of elementary notions of nonstandard analysis

Uno studio sulla *concept-image* di nozioni di base dell'analisi non standard

Mirko Maracci* and Ambra Marzorati
*Dipartimento di Matematica "F. Casorati", Università di Pavia, Italy

Abstract / Nonstandard analysis is a reformulation of mathematical analysis introduced in the 1960s, which allows for extending the system of real numbers so as to include infinitesimal and infinite numbers, and consequently simplify, at least at a first glance, many central notions of elementary calculus. Since then, various proposals have been formulated to introduce the teaching of nonstandard analysis in universities and in upper secondary schools. Even if proponents of such an approach maintain that concepts of nonstandard analysis are closer to intuition and easier to be understood and used, there is not a body of research to support such claims. This paper is meant to contribute to this discussion through reporting on the preliminary results of a pilot-study on student's concept-images in nonstandard analysis.

Keywords: Nonstandard analysis; concept-image; upper secondary school level.

Sunto / L'analisi non standard è una riformulazione dell'analisi matematica introdotta negli anni sessanta, che consente di estendere il sistema di numeri reali in modo da includere numeri infinitesimali e infiniti e di conseguenza semplificare, almeno a prima vista, molte nozioni centrali dell'analisi matematica. Da allora sono state formulate varie proposte per introdurre l'insegnamento dell'analisi non standard nelle università e nelle scuole secondarie. Anche se chi promuove tale approccio sostiene che i concetti di analisi non standard siano più vicini all'intuizione e più facili da comprendere e utilizzare, non esiste un corpus di ricerche che confermi queste ipotesi. Questo articolo intende contribuire a questa discussione presentando i risultati preliminari di uno studio pilota sulle *concept-image* degli studenti in analisi non standard.

Parole chiave: Analisi non standard; *concept-image*; scuola secondaria.

1 Introduction

Nonstandard analysis (NSA) is a reformulation of mathematical analysis introduced in the 1960s by the mathematical logician Abraham Robinson (1965/2013). It develops starting from the introduction of a non-Archimedean extension of the set of real numbers, the so-called set of *hyperreal numbers*, in which one can rigorously define infinitesimals and infinites: infinitesimal numbers are numbers whose absolute value is less than any positive real number, and infinite numbers are numbers whose absolute value is greater than any positive real number.

We can consider it as an extension of real numbers because we can extend to *hyperreal numbers* the arithmetic operations, the order structure and, in general, every function defined on the real numbers, so that every *first order logic statements*[1] 1

1. We can say that «a property P is a first order logic statement when it is expressed by a formula in which every quantifier is restricted on a set. In other words, whenever there is a quantification "for all x..." or "there exists y..." it is necessary to specify in which sets the variables x and y are taking values: "for all $x \in A$..." or "exists $y \in B$...". The quantification on subsets, instead, is not allowed, i.e. formulas containing expressions such as "for all subsets $X \subseteq A$ . . ." are not first order statements» (Di Nasso, 2003, p. 5, translated by the authors).

which is true in the real system is also true in the hyperreal system, and vice versa.[2] 2
For instance, axioms of ordered fields, but also most of the theorems concerning limits, derivatives and integrals, usually taught at the undergraduate level, can be expressed through first-order statements and be valid in both systems.[3] 3

This has remarkable consequences. In fact, within the system of hyperreals, classical standard analysis definitions can be either substituted with, in a sense, "simpler" nonstandard counterparts, or completely avoided; along with the possibility to rely on a rigorous definition of infinites and infinitesimals, this allows for, again, in a sense, "simpler" proofs of nonstandard counterparts of classical standard theorems.

Also for this reason, NSA has raised the interest of logicians and mathematicians both for its applicability in mathematics research and in view of its possible introduction in mathematics teaching curricula. In fact, since its introduction, in various countries (especially in USA) teaching proposals have been formulated, and NSA curricula have begun to be introduced first in some universities and subsequently in upper secondary schools (Keisler, 1976a, 1976b; Harnik, 1986; O'Donovan & Kimber, 1996). Recently, also in Italy, some (a few so far) teachers have started to introduce a NSA curricula in their classes.

One of the reasons for these teaching proposals seems to be the belief that NSA permits to define central concepts in analysis in ways that are closer to the student's *image, spontaneous conceptualisation* and *intuition*.

However there is not a body of research in mathematical education yet that can substantiate such a claim which, rather, has to be problematized and addressed carefully. This paper is meant to report on the preliminary results of a pilot-study (Marzorati, 2018) carried-out among the students of two Italian schools that attended a course on hyperreals and NSA, instead of standard analysis, as part of the class curriculum.

Before presenting the study and discussing the results, in the next section we will briefly illustrate the main features of NSA teaching proposals, and the main reasons for their adoption as part of the class curriculum, that emerge from literature. In the following section then we will discuss an overview of the studies in mathematics teaching related to NSA.

2. The system of hyperreal numbers can be formally introduced via an axiomatic approach or can be built starting from the system of real numbers through sophisticated techniques in Model Theory. Hyperreal numbers, indeed, can be defined as equivalence classes of arbitrary sequences of reals, where the equivalence relation is given in terms of ultraproducts and ultrafilters (Keisler, 1976b).

3. It is worthwhile noting that statements can be formally true in both the systems, but their meaning can be different depending on the system used. Let us consider the Archimedean property: for every couple of positive real numbers x,y, there is a natural number n such that $n \cdot x > y$. This property can be extended to hyperreal numbers; the equivalent statement – for all couples of positive hyperreal numbers x,y there is a ***hypernatural number n*** such that $n \cdot x > y$ – is true in the hyperreal system. However, it is necessary to observe that in this case, n is a ***hypernatural*** number, not a standard natural number and therefore it can be infinite. So, even if formally true, the Archimedean property has a different meaning in this context. For this reason, the set of hyperreals is considered a non-Archimedean extension of the reals. This can also happen in other cases.

2 Main features of NSA teaching proposals

Many NSA teaching proposals in schools are inspired by Keisler's works (1976a) which indeed played a prominent role in bringing Robinson's infinitesimals into the calculus classroom (Kleiner, 2001).
Very briefly, several authors propose an axiomatic approach, by assuming the existence of a non-Archimedean extension of real numbers, that is the existence of a set of numbers, the hyperreals, which includes real numbers (or, better, an isomorphic copy of the real numbers set), infinite numbers and infinitesimal numbers, such that every real function of n real variables is extended in a unique way to a hyperreal function of n hyperreal variables and every n-ary relationship between real numbers is extended in a unique way to a n-ary relationship between hyperreal numbers, and such as that the *Transfer principle* holds.
The transfer principle can be stated as follows:

> «*Transfer principle* (or Leibniz's principle):
> Let $P(a_1,\dots,a_n)$ be a property of standard objects $a_1,\dots,a_n$ expressed as a first order statement. Then $P(a_1,\dots,a_n)$ is true if and only if the same property is true for the corresponding nonstandard extensions, $[a_1^*,\dots,a_n^*]$, i.e. :
> $$P(a_1,\dots,a_n) \Leftrightarrow P(a_1^*,\dots,a_n^*)»$$
> (Di Nasso, 2003, p. 18, translated by the authors)

According to this approach, the issue of the existence or of the possibility to construct such an extension is not addressed in the class, as usually it is not addressed the issue of the existence or the construction of the real numbers' set. To students hyperreals are presented as an "extension" of the real numbers, that are an extension of rational numbers, that are an extension of integer numbers... The process of extending a set of numbers by introducing "new numbers", while keeping some desired formal properties, is familiar for the students, even if it is rarely treated as problematic or accompanied with an explicit discussion.
In order to make infinites and infinitesimals more "visible", many authors introduce imaginary optic devices: infinitesimal microscopes and infinite telescopes, that depending on the scale factor (finite, infinite or infinitesimal) can visualise respectively hyperreal numbers (Figure 1).

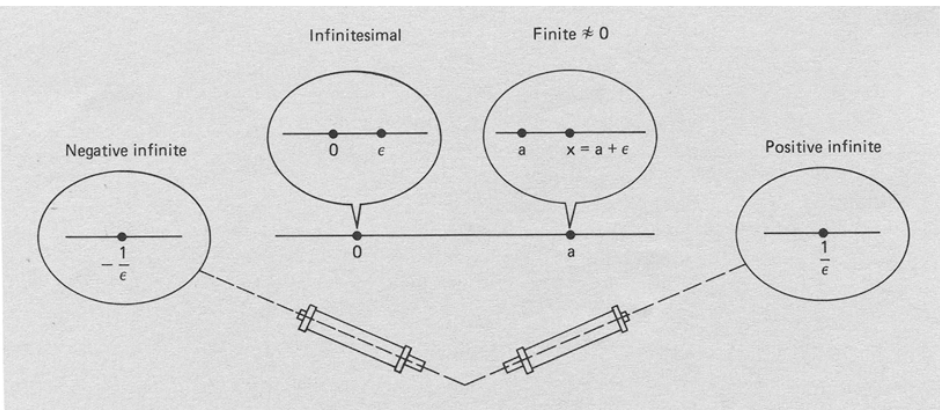

Figure 1
Infinitesimal microscopes with scale factor $\frac{1}{\varepsilon}$ centred in *0* and in a real number a and infinite telescopes pointing at $\frac{1}{\varepsilon}$ and $-\frac{1}{\varepsilon}$ (retrieved from Harnik, 1986, p. 45).

Central in NSA are the relations of *infinitely close, indistinguishable* and the *standard part function*; starting from these notions it is possible to reformulate in NSA the definitions of the main notion of standard analysis.

Definition (*infinitely close*)*:* two hyperreal numbers are *infinitely close* if and only if their difference is an infinitesimal.
Definition (*indistinguishable*): two hyperreal numbers are *indistinguishable* if their ratio is infinitely close to 1.

These two relations are equivalence relations. The terms *infinitely close* and *indistinguishable* can evoke similar intuitive meanings, but the two notions are different. For example, if ε is an infinitesimal, ε and ε^2 are infinitely close, but not indistinguishable, and if M is an infinite and a is a finite (but not infinitesimal), $M+a$ e M are not infinitely close but are indistinguishable.
It is possible to prove that, given a hyperreal finite number, there exists a unique real number infinitely close to it. This is used to define the standard part of a finite hyperreal number.

Definition (*standard part*)*:* the *standard part of a finite hyperreal* is the unique real number which is infinitely close to it.

Drawing on these notions it is possible to define the notions of limit (even if some teachers do not present limits, because they are not necessary for constructing the NSA system), continuity, derivative and integrals, without recurring to $\varepsilon - \delta$ formalism. For instance, let f be a real function of one real variable and x_0 an accumulation point of the domain of f. The limit of f as x approaches x_0 can be defined as follows:

Definition: we say that the limit of f as x approaches x_0 is finite if there is a real number L such that whenever x is infinitely close to x_0 but different from x_0, $f(x)$ is infinitely close to L.

Let us notice that if x is infinitely close to x_0 then x_0 is an hyperreal number, but not a real one, so the symbol *"f"* represents both a real function of one real variable and its hyperreal extension. In fact, $f(x)$ must be a hyperreal number, otherwise it would not be possible for it to be infinitely close to L. In this definition, for sake of readability, there is not a distinction between a real function f and its extension f^*.

The last definition we want to show is that of the tangent line to the graph of a function at a point. In NSA it is defined as the line passing through two infinitely close points of the graph of the function (Figure 2); and it is usually introduced together with the notion of derivative, defined as the standard part of the slope of that line.

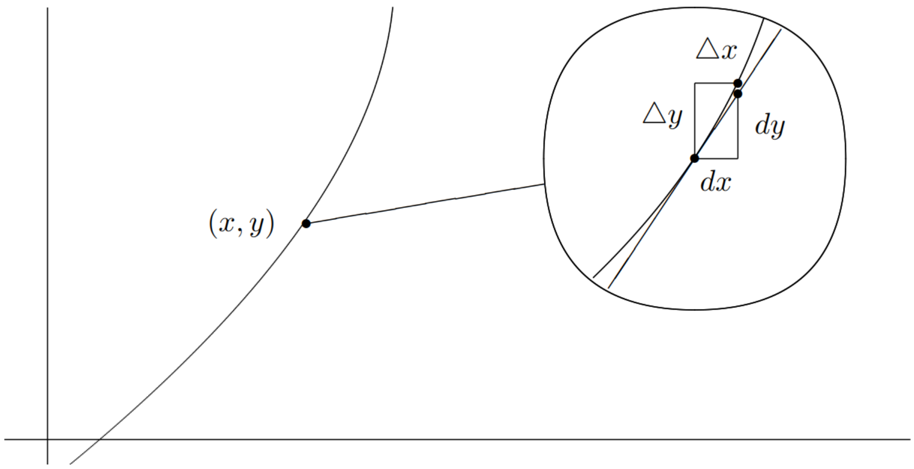

Figure 2
Infinitesimal microscope with scale factor $\frac{1}{dx}$ centred in (x,y) (retrieved from Keisler, 1976b, p. 37).

Once the derivative is introduced it is possible to state and prove the increment theorem. This theorem allows us to give a precise meaning to the statement that the tangent line to a function graph at a point and the function graph itself are indistinguishable for every point infinitely close to the tangency point.

Theorem (*increment*). Let f be a differentiable function and dx an infinitesimal, different from 0. Then there exists an infinitesimal $\varepsilon \neq 0$ such that $f(x+dx)-f(x)=f'(x)dx+\varepsilon dx$.

As we said before, the definitions in NSA appear to be "easier" and lighter due to the absence of the usual formalization. Also, they use evocative terms defined in a rigorous way, like "infinitely close" and "indistinguishable".

We end here our brief presentation of the features of NSA teaching proposals. Our aim was to give an idea of them to allow the reader to understand the reasons why some teachers decided to use this approach and to be able to follow the discussion in our study. The reader interested in exploring this subject can find some hints in the list of references.

2.1 The reasons behind the choice of a NSA approach

Keisler (1976a; 1976b) advocates three main reasons in favour of the teaching of NSA: to provide students with extra mathematical tools with possible important applications in the future; to make central concepts, such as derivative and integral, easier for students to understand and apply; and, above all, to introduce concepts closer to students' intuition. These arguments are generally shared by proponents of such an approach, for instance Machover (1993, p. 207) states that «NSA is a very powerful tool: it makes many mathematical concepts much more intuitive; and nonstandard proofs are often shorter, easier and more "natural" than their standard counterparts».
Harnik observes also that since NSA gives explicit and rigorous definitions of expressions like "infinitely small", usually used in a figurative way, then it is possible to use them in a formal mathematical discourse. «At the classroom level, the main importance of Robinson's contribution is that it reassures us, the teachers, that when we say "infinitesimal", we can finally claim that we know what we are talking about» (Harnik, 1986, p. 63).

Lastly, there are also those who see in NSA the rightful completion of a historical process, that lasted centuries and gives dignity, in a mathematical sense, to the ideas of infinitely small and infinitely large which characterize analysis and need to be known by teacher and students. «For over two millennia infinitesimal methods have been used with great success by such as Archimedes, Leibniz, Newton, Euler, and Cauchy. Robinson's nonstandard analysis is a fitting culmination, if not a vindication, of these ideas» (Kleiner, 2001, p. 172).

In our study we interviewed five teachers that have been teaching NSA for at least 3 years, in different schools in different cities[4]. Our objective was to know the reasons why these teachers chose to include NSA in their curriculum, and what are in their opinion the potential and critical points of NSA teaching. We will not discuss this in details, but it is interesting to notice that, concerning the reasons in favour and the potential of a NSA approach, we have a full correspondence between the teachers' opinion and what can be found in literature. It is generally a bit worrying to identify the adequacy of the level of rigour when introducing hyperreal numbers, and the need to introduce in a short time new relations that are not so easy to distinguish. 4

Though the above arguments might appear reasonable, there is not a body of research which can confirm in a convincing way that concepts of NSA are closer to students' intuition (whatever is meant by intuition) than the concepts of standard analysis. On the contrary the amount of research on the NSA teaching and learning is rather poor.

3 Research on students' learning of basic concept in NSA

As mentioned above the proposal of teaching NSA instead of standard analysis does not seem to have raised a discussion within the mathematics education research community up to now.

One exception is constituted by the study carried out in the 1970s by Kathleen Sullivan (1976). In her study Sullivan proposed to a group of teachers with experience in the teaching of standard analysis to implement a teaching intervention based on Keisler's work (1976a).

The study involved 68 students from 5 different schools. In order to evaluate teaching results, Sullivan gave a purposefully designed calculus test to the experimental group and to a control group and compared their answers. The questions were intended to test the «ability of the students to define basic concepts, compute limits, produce proofs, and apply basic concepts» (Sullivan, 1976, p. 372). Even if there were some interesting differences between the performances of the two groups, the conclusions drawn were rather cautious: on the one hand, NSA can be a viable alternate approach to teaching calculus, but on the other one, «this is not "Calculus made easy"» (Sullivan, 1976, p. 375). In particular, no findings supported the possible intuitiveness of NSA concepts.

4. In the school year 2017/18 the 5 teachers were respectively working in a *liceo delle scienze umane, opzione economico-sociale* in Verona, in an *istituto tecnico industriale in Modena*, in an *istituto tecnico per geometri in Morbegno* (SO), in a *liceo classico* in Venezia and in a *liceo scientifico* in Milano, and had taught NSA respectively for 3, 7, 5, 18 and 7 years.

On the other side, studies in standard analysis reveal that students can develop intuitions or beliefs which might appear to be resonant with nonstandard concepts. That might happen for instance with the concept of limit, when students are exposed to expressions such as "getting a variable arbitrarily small" or "arbitrarily large" (Tall, 1990; Cornu, 1991). Besides surface similarities, «even though the informal use of infinitesimals may seem to be closer to nonstandard analysis, students' spontaneous beliefs are often inconsistent with nonstandard theory too» (Tall, 1993, p. 18).
A remarkable exception seems to be that of Sarah, an undergraduate calculus student, who, in the context of a study on students' conceptions in calculus (Ely, 2010), revealed «robust conceptions of the real number line that include infinitesimal and infinite quantities and distances» (p. 117), sharing similarities with how these notions are formalized in NSA.

4 Our research project

This paper reports on the preliminary results of a pilot-study whose main object is to investigate students' conceptions[5] in NSA in order to discuss, possibly confirm or reject, the claim at the basis of many proposals supporting the teaching of NSA according to which concepts of NSA are "closer to intuition" than standard analysis ones. 5
The nature of the research problem is thus very open and exploratory; on the other hand, the scarcity of the existing literature on this topic does not allow for a sharper perspective. To preserve openness we decided to frame our study within the broad theory of *concept-image* and *concept-definition* developed by Tall and Vinner (1981) and largely used in mathematics education research.

4.1 *Concept-image* and *concept-definition*

In education, it is widely acknowledged the need to discern between the definition of a notion in a knowledge system and the personal sense assigned to it by an individual. In the research field of mathematics education, an important reference is represented by the works of Tall and Vinner, who propose to discern between *concept-definition* and *concept-image*.
The expression *concept-image* has been introduced to account for the non-verbal dimension – constituted by visual representations, impressions, experiences – which is associated in one's mind with the definition of a concept. Thus the expression *concept-image* describes «the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes» (Tall & Vinner, 1981, p. 152).
The *concept-image* can change over time according to the information progressively acquired by the student and does not necessarily have a coherent structure: different aspects, eventually in conflict, can emerge in different moments or contexts.
The expression *concept-definition* denotes a linguistic entity: a word or a combina-

5. We use the term conception after its common use in mathematics education to refer to the existence of a plurality of different points of view on the same mathematical object, and distinguish the suitability of these points of view in solving a certain kind of problems (Artigue, 1991).

tion of words and symbols used to specify a concept (Tall & Vinner, 1981). It can be formal or personal and can change from time to time.
When accomplishing a task, *concept-image* and *concept-definition* are ideally supposed to interact in a synergy, for example as pictured in Figure 3. In this picture *concept-definition* and *concept-image* regarding the same notion are represented with two distinct boxes. The arrows show that, in the depicted case, the resolution process starts with the use of the concept-definition, activated by the given problem, and develops through an interaction between *concept-definition* and *concept-image*. The result is an answer controlled by the *concept-definition*.

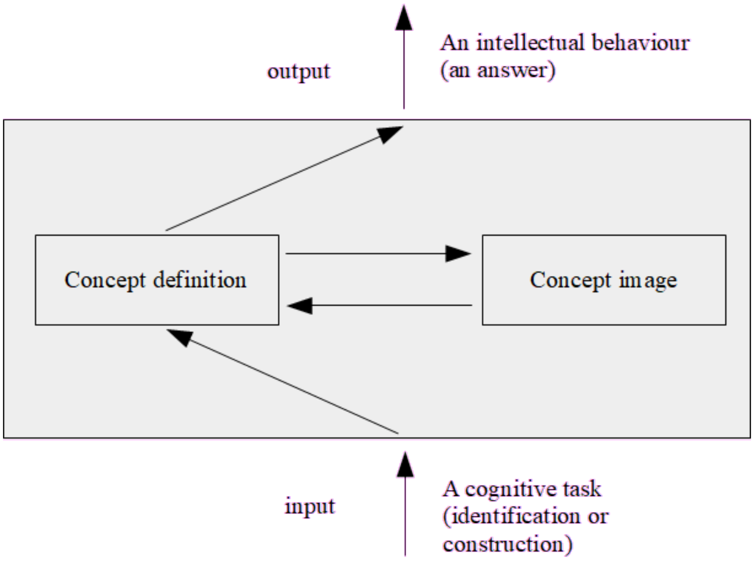

Figure 3
Ideal interplay between *concept-image* and *concept-definition* (retrieved from Vinner, 1991, p. 71).

But this is just an ideal situation; the dynamics between *concept-image* and *concept-definition* may assume different forms. One can use only the *concept-definition* when required to give an explicit definition or use only the *concept-image* when the definition is not required. This is the case of intuitive responses: the student solves a problem, or answers a question without consulting the *concept-definition* (Figure 4).

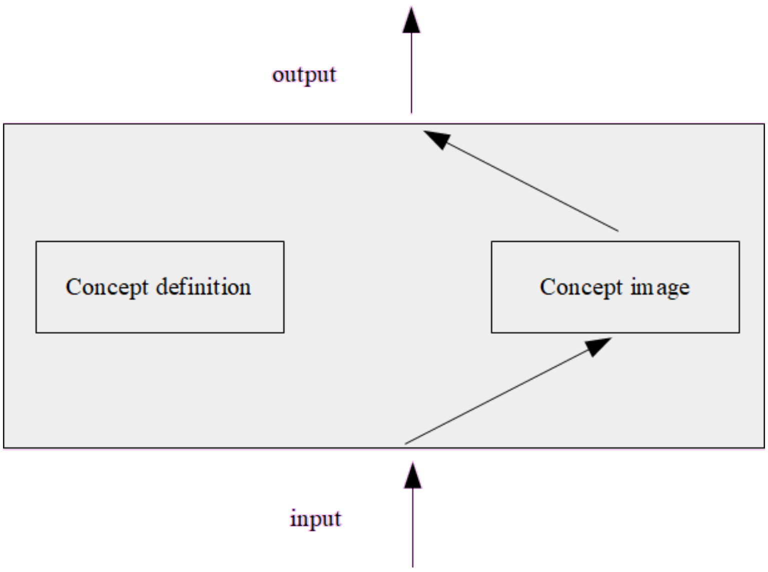

Figure 4
Lack of interaction between *concept-image* and *concept-definition* in the case of intuitive responses (retrieved from Vinner, 1991, p. 73).

This does not mean that an intuitive response is necessarily wrong, just that the answer is not subjected to the explicit control by the definitions in play. Problems arise when the *concept-image* includes parts that are not consistent with the mathematical theory at stake, and *concept-image* and *concept-definition* are in conflict.

4.2 Research context and participants

The study took place during the school year 2017/2018 and involved 76 students of 4 different classes of 2 Italian upper secondary schools[6]: 3 classes (namely, 11th, 12th, 13th grades) of a *liceo delle scienze umane* in Verona, and a 12th grade class of an *istituto tecnico industriale* in Modena. 6

The 11th and 12th grade students from Verona attended a course of introduction to the hyperreal system during the school year 2017/2018; the 13th grade students from Verona studied hyperreals in a previous year and during the school year 2017/2018 attended a NSA course that included the main definitions, theorems and techniques of NSA; the students from Modena attended a course including both hyperreals and central concepts in NSA during the school year 2017/2018. These courses were part of the usual curriculum designed and enacted by their class teacher.

At the end of the school year we gave the students a questionnaire to be answered in 45 minutes. We will describe it in the next section.

4.3 The questionnaire: global structure and tasks

In this paper we present and discuss the results obtained through the questionnaires given to the 76 students involved in the study. Since the students attended different courses with different curricula, we decided to assemble 4 different questionnaires, one for each class, starting from the same corpus of tasks with some common aspects. We designed a list of tasks, with both open and closed questions, concerning different aspects of NSA. More precisely, we chose to inquire students' *concept-images* related to diverse aspects of NSA.

I. *The nature of hyperreals.* The aim of this group of tasks is to find out if the presentation, the examples and the introductory problems proposed to the students are sufficient to create a meaningful context for the introduction of the hyperreals.
II. *The notions of infinitely close, indistinguishable and standard part.* The aim of the tasks in this group is to inquire the conceptions regarding the notions of infinitely close, indistinguishable and standard part.[7] 7
III. *Order structure and field structure.* These tasks address students' concept-images on order relations, in particular the comparison between infinites, and infinitesimals and the extension of field operations (sum and product) to infinites and infinitesimals.

6. Italian upper secondary school (*scuola secondaria di secondo grado*) corresponds to the 4th year of *scuola media* and to the *scuola media superiore* in *Canton Ticino*.
7. From a didactic point of view, we think that the close introduction of different notions, such as infinitely close and indistinguishable, that denote aspects that might seem similar at the students' eyes, can be confusing. We also think that the concept of infinitely close might be interpreted as "very, very close", so with a meaning that differs from the formal one. With this interpretation, numbers like 0,0000001 and 0,0000002 can be considered very close, but they are not infinitely close.

IV. *The notion of limit, tangent line to the graph of a function and integral.* Among the reasons supporting NSA, there is the belief that the main notions of analysis are easier, and simpler to be addressed in an operative way; these tasks are designed to inquire this hypothesis.

The above-mentioned aspects are intertwined, but we tried to design tasks that addressed mainly one of them, in order to better identify and discuss possible critical issues related to them.

Starting from these tasks we assembled 4 different questionnaires – one for each class – so as to cover as many aspects as possible and to try to have common tasks in every questionnaire. The number of tasks varies between 11 and 13. The main difference is the portion of curriculum covered. We discussed the appropriateness of the tasks with the teacher of each class so that they could evaluate their consistence with the lessons and the didactic aims.

For every task, except for I.1, students were also asked to make clear their level of confidence in the correctness of their answer. They could choose one among three options (**Figure 5**).

In the following tasks you may find emoticons near the answer options. They allow you to denote the level of confidence you have in your answer:

 I am highly confident

 I am quite confident

 I am not confident at all

For every answer cross off the emoticon that better represents your level of confidence.

Figure 5
Markers for the level of confidence in the answer.

5 Findings

Given the exploratory nature of the study and the differences among the questionnaires, it is difficult to give a unifying synthesis, so we decided to provide a qualitative discussion of the results. We will focus on the tasks that seem to give the most interesting cues.
To give a complete view of the study we will discuss at least one task for each aspect: I) the nature of the hyperreals; II) the notion of *infinitely close, indistinguishable* and *standard part*; III) order and field structure; IV) the notions of limit, tangent line to the graph of a function, and integral.
The tasks are identified with a label made by: a roman numeral to indicate which aspect, among those mentioned in the previous section, is related to the task (I, II, III, or IV); a number to distinguish the tasks relating to the same aspect; a number (11, 12, or 13) to indicate the grade of the class that answered that task; a combination

of letters for the city (MO for Modena, VR for Verona).
In the text of the questionnaire given to the students the tasks were numbered progressively.

5.1 On the nature of hyperreals and why they are introduced

The task I.1 (Figure 6) appears in all the questionnaires. We ask the students to explain why, in their opinion, the hyperreals are introduced in mathematics. Mathematical objects can be considered conceptual tools for particular aims (for instance to solve theoretical and practical problems), thus having an idea of the problematic context in which they can be used is important to understand them. With this task we wanted to inquire the meaning that students give to their activities with hyperreals: are the proposed activities effective enough to give the students a context of meaning for these new numbers? If this context does not coincide with the one proposed by the teacher, which context of meaning did the students associate to the hyperreal numbers? Given the nature of this specific question, it would not have made any sense for us to classify the answers as right or wrong. Also the request to specify the level of confidence in the answer did not seem significant; on the contrary, it could have been a possible source of confusion for the students.

Task N°1 In your opinion, why are hyperreal numbers introduced?

Mark to what extent you agree with the following reasons.

1: I totally disagree

2: I partially disagree

3: I am indifferent

4: I pretty much agree

5: I totally agree

	1	2	3	4	5
There are points on the straight line with no correspondence with any real number, so we introduce hyperreal numbers to denote such points					
Hyperreal numbers are introduced to solve practical problem, such as finding the tangent line to the parabola, or computing instantaneous velocity					
Hyperreal numbers are introduced to denote very small and very large quantities.					
Hyperreal numbers are introduced to justify the Eudoxus-Archimedes' axiom					
Hyperreal numbers are fictitious numbers, introduced to solve problems which will be addressed in the future					

Can you think of other reasons?

Figure 6
Task I.1 – 11VR, 12VR, 13VR, 12MO.

The picture emerging from the answers is complex, but it is possible to detect interesting pieces of information. Let us focus our discussion on items A, C, and D, which are the most connected to the passage from real to hyperreal numbers.[8] 8

Table 1 presents the percentages of the students' answers related to such items: we grouped the answers "I totally agree" and "I pretty much agree" on the unique line "Agree", and the answers "I totally disagree" and "I partially disagree" on the unique line "Disagree". In this table the percentages related to both the void answers and the option 3 ("I am indifferent") are not considered.

Item	Agreement level	11VR (24)	12VR (21)	12MO (18)	13VR (13)
A	Agree (4,5)	62,50%	66,67%	38,89%	23,08%
	Disagree (1,2)	8,33%	4,76%	44,44%	61,54%
C	Agree (4,5)	79,17%	80,95%	61,11%	100%
	Disagree (1,2)	8,33%	19,05%	22,22%	0,00%
D	Agree (4,5)	25%	19,05%	27,78%	38,46%
	Disagree (1,2)	58,33%	61,90%	44,44%	23,08%

Table 1
Percentages of answers – Task I.1.

Even if there are many differences from class to class, a high percentage of students agrees with statement C, which states that hyperreal numbers are introduced to denote large and small quantities. This result is not easy to interpret, since we do not know what meaning was given by the students to the expression "denote quantities", which can evoke practical contexts, but can also be used in abstract contexts, in a metaphoric sense. However, the high percentage of students agreeing with this statement points out a possible criticality in the way hyperreals are presented. If we consider the answers to the other items these criticalities seem even stronger. Regarding item A, it is interesting to note that there is a strong difference among the classes: on the one hand we have the students of 11VR and 12VR, for whom the introduction of hyperreals is more recent, and on the other hand we find the students of 12MO and 13VR, where the introduction of the hyperreals is less recent. This difference might suggest the possibility that the introduction of the hyperreals undermines at the beginning the role of the geometric line as a possible representation for the real numbers.

The percentage of students who believe that hyperreals are needed to justify the Archimedean property (statement D) is not very high, but it is not negligible at all: it seems to point out students' disorientation and lack of awareness about the fundamental properties of reals and hyperreals and the reasons behind the introduction of the latter. If we consider only the classes 11VR, 12VR, 13VR, that shared the same teacher, it would seem that such a disorientation is more frequent among the stu-

8. The complete tables, with the percentages for every option of this and other tasks, can be found in the Attachment 1.

dents of 13VR, which on the one hand are those for whom the introduction of hyperreals is most distant in time but on the other hand are also those who have worked for the longest time with this number system.
The majority of students seem to consider the system of real numbers inadequate to cope with large or small quantities or to denote all the points on a line. That may be due to the way hyperreals were introduced in these classes: making use of example in which very large and very small quantities are compared between them to introduce the idea of a quantity which is negligible with respect to another; or of pictorial devices which can suggest that there are point on the line which are not covered by reals. In any case, the students' understanding of real numbers is called into question. The *concept-images* of both reals and hyperreals which emerge from these answers appear inadequate from a mathematical point of view. The effort of the teachers to introduce hyperreals appealing to intuition does not seem successful. This leads to wonder whether the introduction of hyperreals can be successful if students do not develop an appropriate *concept-image* of the system of real numbers.

5.2 On the notions of infinitely close, indistinguishable and standard part

These notions do not have a standard counterpart, but they are fundamental in NSA for the definitions of limit, derivative and integral.
We will examine 3 tasks: the first 2 (Figure 7 and Figure 8) are in the questionnaires given to the classes 11VR, 12VR, 12MO; the last one (Figure 9) was included in all 4 questionnaires.

Task N°2 If a is a hyperreal finite number, but not a real number, then

a is infinitely close to a unique real number	T	F			
a is infinitely close to exactly two real numbers	T	F			
a can be infinitely close to several real numbers	T	F			
a can be infinitely close to an infinite number	T	F			

Figure 7
Task II.1 – 11VR, 12VR, 12MO.

In the task in Figure 7, the only true statement is the first one, the other three are false. As we can see from Table 2, the percentage of correct answers is high, except for the students of 11VR.

Item	Option	11VR (24)	12VR (21)	12MO (18)
A	**True (right answer)**	**45,83%**	**85,71%**	**94,44%**
	False	54,17%	14,29%	5,56%
B	True	33,33%	4,76%	0%
	False (right answer)	**66,67%**	**90,48%**	**100%**
C	True	25%	14,29%	5,56%
	False (right answer)	**75%**	**80,95%**	**94,44%**
D	True	29,17%	28,57%	5,56%
	False (right answer)	**70,83%**	**71,43%**	**94,44%**

Table 2
Percentages of answers – Task II.1.

Less than half of the students of 11VR answered correctly the item A. We cannot provide any specific interpretation for that without further investigations. We acknowledge that some students could have disregarded that *a* is a finite hyperreal number (if *a* were not finite, A would be false and D true). But even in that case, the reasons why this happened at such an extent only in 11VR class and not in the other ones, remain unexplained. Yet, the global outcome might appear positive in itself; but if we consider also the answers to other questions of this group the whole picture is not that encouraging. Let us consider the task II.2 (**Figure 8**).

Task N°5 π is a number with an unlimited decimal expansion. Let us take an approximation: 3,1415. Can we say that 3,1415 is infinitely close to π?

YES NO

Figure 8
Task II.2 – 11VR, 12VR, 12MO.

The right answer is no: π and 3,1415 are two distinct real numbers, their difference is still a real number and so they are not infinitely close. This is a general property directly linked to the content addressed in the previous task (II.1): no number can be infinitely close to two real numbers, since in turn they cannot be infinitely close to each other.

9

Option	11VR (24)	12VR (21)	12MO (18)	Level of confidence[9]		
				High	Medium	Low
Yes	79,17%	80,95%	38,89%	48,84%	37,21%	9,3%
No	**20,83%**	**19,05%**	**61,11%**	40%	45%	10%

Table 3
Percentages of answers – Task II.2.

9. The percentages of the level of confidence in the two rows are computed with respect to the total number of students who answered "Yes" (43 students) or respectively "No" (20 students) to the task. The two sums are less than 100% because some of the students who answered the task did not express their level of confidence.

Students could have easily answered this question, if they had used the formal definition. This remark leads us to suppose that students relied on *concept-images* of the relation of “being infinitely close” and of the notion of infinitesimal as “very small” or “arbitrarily small” number. Such *concept-images* do not prevent from answering correctly the first task, but are not adequate in this context. Also the *concept-images* of real number, decimal expansion, approximation etc. appear inadequate in this case. It is worthwhile noticing that 85% of the students declares to be highly or quite confident about their answer; this percentage is surprisingly high considering how cautious students usually are when expressing their level of confidence about their own answers in mathematics. This suggests that such concept-images are deeply rooted and may be hard to destabilize.

Task N°7 ε is an infinitesimal, different from 0. In the following table, cross off the corresponding cells, if $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ is equal, infinitely close or indistinguishable from $1, \frac{1}{\varepsilon}, \frac{1}{2}, \frac{1}{2\varepsilon}$.

	Is equal to	Is infinitely close to	Is indistinguishable from	None of the previous answers		
$\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$					1	
					$\frac{1}{\varepsilon}$	
					$\frac{1}{2}$	
					$\frac{1}{2\varepsilon}$	

Figure 9
Task II.3 – 11VR, 12VR, 12MO, 13VR.

The only true relation is that $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ is indistinguishable from $\frac{1}{2\varepsilon}$, while it is neither equal, nor infinitely close to, nor indistinguishable from 1, $\frac{1}{2}$ or $\frac{1}{\varepsilon}$. It should be noted that, in principle, the option “equals to”, “infinitely close to” and “indistinguishable from” are not mutually exclusive: for instance, two non-infinitesimal, infinitely close numbers are also indistinguishable. The answers highly vary according to the specific relation considered and from class to class; in the **Attachment 1** one can find the complete survey of the percentages of answer. Here we focus on those concerning the possible relations between $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ and $\frac{1}{2\varepsilon}$ (**Table 4**), which offer hints for more general considerations.

$\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ and $\frac{1}{2\varepsilon}$ are:	11VR (24)	12VR (21)	12MO (18)	13VR (13)
Equal	45,83%	28,57%	5,56%	53,85%
Infinitely close	12,50%	19,05%	5,56%	7,69%
Indistinguishable	**29,17%**	**52,38%**	**83,33%**	**30,77%**
None of the previous answers	4,17%	4,76%	5,56%	7,69%

Table 4
Percentages of answers - Task II.3.

The percentage of correct answers varies from class to class: it goes from 29,71% in 11VR class to 83,33% in 12MO class, which exceeds the other classes for this specific item. What can be surprising is not only the high number of students of the classes in Verona who do not realize that $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ and $\frac{1}{2\varepsilon}$ are indistinguishable, but also the high percentages of those who wrote that the two numbers are actually equal: 45,83%, 28,57% and 53,85%; by the way, if two number are equal they are also infinitely close and indistinguishable.

In order to correctly answer this task, one can recall the definition of the relations at stake and then make the needed calculations. For instance, in order to ascertain whether $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}$ and $\frac{1}{2\varepsilon}$ are infinitely close one needs to establish whether $\frac{1+2\varepsilon+\varepsilon^2}{2\varepsilon}-\frac{1}{2\varepsilon}$ is an infinitesimal or not. After a few calculations, in a couple of steps, one finds that the above number is equal to $1+\frac{\varepsilon}{2}$ which is not an infinitesimal.

The wrong answers can be due both to an inadequate *concept-image* which is strong enough to inhibit any resort to the formal definition, and to difficulties in executing the calculations required (which, however, are formally equivalent to usual calculations with algebraic fractions). With respect to this latter remark, in general when dealing with hyperreals, according to the aim, one can formally simplify the calculations by suitably disregarding infinitesimals (as for instance when computing the standard part of a finite hyperreal number). This way of making calculations can be partly automatized but sometimes demands a strong conceptual control. If such a control is weak or completely absent, the risk of making errors is high. All that contrasts with the supposed greater ease of making calculations and formal manipulations with hyperreals, that in turn, according to some authors, could make easier to prove theorem in NSA.

Also in other tasks of this group, students' wrong answers can be interpreted as the effect of a missing or weak conceptual control over calculations with hyperreals.

5.3 On the order and field structures of the set of hyperreals

Several tasks of the previous group involve properties of the set of hyperreals which are related to either its order or its field structure, even though these properties were not the major focus of those tasks. Also for this reason, in order to avoid useless repetitions, we discuss the outcomes of a single task of this group (Figure 10).

Task N°6 In your opinion, if I add a positive infinite number to itself:

I get the same infinite number	T	F
I get a different infinite number, but infinitely close	T	F
I get a different infinite number, of the same order, but at an infinite distance	T	F
I get a different infinite number of a higher order	T	F

Figure 10
Task III.2 – 11VR, 12VR.

The 4 statements are mutually exclusive, and the only true statement is the third one. They are nevertheless presented as independent from each other, in order to track down possible contradictory elements in the students’ *concept-images*. We could have formulated the task in a different way, by asking students to compare numbers M and $2M$, or M and $M+M$, being M an infinite number; we chose to formulate the task using words rather than symbols in order not to orient students towards a necessarily symbolic or algebraic treatment of the task. We left to the students the responsibility to make decisions about the most suitable way to address the task.

Item	Option	11VR (24)	12VR (21)
A	True	62,50%	19,05%
	False (right answer)	**37,50%**	**80,95%**
B	True	33,33%	23,81%
	False (right answer)	**66.67%**	**76,19%**
C	**True (right answer)**	**50,00%**	**57,14%**
	False	50,00%	42,86%
D	True	62,50%	28,57%
	False (right answer)	**37,50%**	**71,43%**

Table 5
Percentages of answers – Task III.2.

When we designed this task, we figured out that some students could have given contradictory answers. This actually happened. For each statement, the percentage of right answers is generally quite low; but the most interesting issue here is that 26 out of 45 students indicate contradictory statements as simultaneously true. In fact:

– according to 13 students the sum of an infinite number with itself is both equal and different from the starting number (combination of items A-C, A-D, A-C-D true);
– according to 6 students the order of infinity of the outcome is both the same and different from the order of infinity of the starting number (items C-D true);

– at last, according to 13 students the outcome is both infinitely close and infinitely far from the starting number (items B-C, B-D, B-C-D true).

Only 19 students gave consistent answers (indicating a single true statement), and among them only 10 students answered correctly by indicating the statement C as the only true statement.
The answers to this question therefore confirm that the students' *concept-images* can be in contradiction with the mathematical theory in certain situations, but also present internal contradictory elements. Finally, we note that the level of confidence expressed by the students for these answers is generally medium-low, further confirming their disorientation.

5.4 On the notions of limit, tangent line and integral

We designed some tasks with the aim of investigating whether the definitions in NSA of some basic notions of standard analysis might result more intuitive and easier to deal with for students. These tasks are included only in the questionnaires for the 12MO and 13VR classes which covered this part of the curriculum; the formulations of some tasks, such as task IV.1, are different for the two classes since the respective teachers presented these notions in different ways. The version of the task IV.1 for 13VR refers explicitly to the notion of limit (IV.1a, Figure 11), while the version for 12MO refers to the notion of asymptote (IV.1b, Figure 12).

Task N°6 About a function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ you know that, if M is a positive infinite, $f(M) = 1$ and $f(2\mathrm{M}) = -1$. What can you say about $\lim_{x\to+\infty} f(x)$?

Choose an option:

A) I cannot say anything, it depends on what happens with other infinite numbers

B) The limit does not exist because the function has different values for different infinite numbers

C) The limit is 0 because the function keeps changing from 1 to -1

D) Else (explain)

Figure 11
Task IV.1a – 13VR.

Task N°9 About a function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ you know that, if M is a positive infinite, $f(M) = 1$ and $f(2M) = -1$. What can you say about the behaviour at infinity of the function?

Choose an option:

A) I cannot say anything, it depends on what happens with other infinite numbers

B) The function does not have a stable behaviour at infinity because it takes different values for different infinite numbers

C) The function has an asymptote in 0 because it keeps changing from 1 to -1

D) Else (explain)

Figure 12
Task IV.1b – 12MO.

The right option for both the versions is B. As a matter of fact, with respect to the version IV.1a (**Figure 11**), a function f has a finite limit as x tends to $+\infty$ if and only if the image of any positive infinite by f is always a finite number and its standard part does not depend on any specific infinite; the limit is $+\infty$ (resp. $-\infty$) if and only if the image of any positive infinite is always a positive (resp. negative) infinite. Hence one can claim that the limit of f as x tends to $+\infty$ does not exist. Analogously, as for the version IV.1b (**Figure 12**), one can claim that f does not have any horizontal asymptote. The formulation of the option B could be ambiguous indeed since it refers to the idea of stable or unstable behaviour.

Table 6
Percentages of answers – Tasks IV.1a and IV.1b.

Option	IV.1b – 12MO (18)		IV.1a – 13VR (13)	
A	5	27,78%	5	38,46%
B	**10**	**55,56%**	**3**	**23,08%**
C	1	5,56%	2	15,38%
D	2	11,11%	1	7,69%

As we can read in **Table 6**, the number of students who faced this group of tasks is low; hence no general conclusion can be drawn. We can only observe that, for both the versions, we gathered a low number of correct answers (very low in 13VR class). That suggests the following reflection. NSA provides operational means for the explicit calculation of limits that are at the same time rigorous and easier than those of standard analysis. Notwithstanding, this does not lead necessarily to the development of *concept-images* which are more adequate from the mathematical point of view. Hence, when no explicit calculation is asked or possible, students meet difficulties that are similar to those which are documented in the teaching and learning of standard analysis.
Once again, the students' level of confidence is medium-low as to testify their disorientation.

The tasks IV.2 and IV.3 (**Figure 13** and **Figure 14**) investigate the *concept-images* related to the notion of tangent line to the graph of a function at a point.

Figure 13
Task IV.2 – 12MO, 13VR.

Task N°11 How would you explain what is the tangent line to the graph of a function at a point?

The task IV.2 does not ask explicitly for a formal definition of the tangent line to the graph of a function at a point: in fact we did not intend to foster students' recourse to such definition, but rather to promote their effort to make explicit their personal sense of the concept of tangent line. It is known from literature (e.g., Biza, 2007; Vinner, 1983) that students ascribe to the tangent line to a graph several features which may be in conflict with the formal definition, such as: the tangent line touches the graph at a single point, it does not cut the graph (neither locally nor globally), it cannot overlap with the graph even for a short distance, or it always exists. One of

the aims of the task IV.2 is to reveal the possible presence of these elements in the *concept-images* of NSA students.
The students' answers are various, nevertheless there are some common traits, which are especially relevant for the present study:

- 14 students out of 31 mention that the tangent line touches the graph at a single point (these answers are labelled *point* in the Table 7 below).
- 7 students out of 31 try to relate explicitly the slope of the tangent line with the derivative of the function at a point, some of them express the relation using words, others use also symbolic expressions (see label *derivative* below).
- 5 students out of 31 mention explicitly that the tangent line is indistinguishable from the graph of the function near the point of tangency (see label *indistinguishable* below).

Five students, already cited above, refer explicitly to more than one of these features, e.g.: «It is a line that touches the function at a point and is indistinguishable from it».
Seven students do not provide any possible definition or description of what they mean by tangent line (some of them just sketch graphically a tangent line to a function graph, others not even that).
The answers of the remaining students do not refer to any of the aspects mentioned above, and it is difficult to find common traits among them; indeed some answers are hard to interpret, as for instance: «The tangent line is perpendicular to the function».
It is surprising the low number of those who mention that the tangent line and the function graph are locally indistinguishable, since this relation is both rigorously defined in NSA and apparently highly evocative.

Label	13VR (13)		12MO (18)	
Point	7	87,50%	7	43,75%
Derivative	0	0,00%	7	43,75%
Indistinguishable	0	0,00%	5	31,25%
Other	3	37,50%	2	12,50%
Missing/sketch	5		2	

Table 7
Synthesis of the answers – Task IV.2.[10]

10

Obviously the direct question «How would you explain what is the tangent line to the graph of a function at a point?» is not enough to access an individual's *concept-image*. Firstly, in a situation linked to the school experience, a student can be led to answer in a way that satisfies what she/he believes to be the teacher's expectations, even if it is not the teacher who asks the question. Secondly, different elements of an individual's *concept-image* may emerge in different situations. Lastly an individual is not generally aware of her/his *concept-image* related to a certain concept, in all its aspects, therefore she/he could not completely express it by words.

10. Since some answers refer to different aspects and some students did not answer, the percentages are calculated on the answers with an attempt of verbal or symbolic description. The sum of the percentages exceeds 100% because sometimes an answer refers to more than one aspect.

For this reason, we included another task investigating the *concept-image* of tangent line: the task IV.3 (Figure 14). This task asks to recognize which diagrams may represent a function and the tangent line to its graph at a point regardless the explicit formulation of its features. The task is adapted from the task used by Biza in her research (2007).

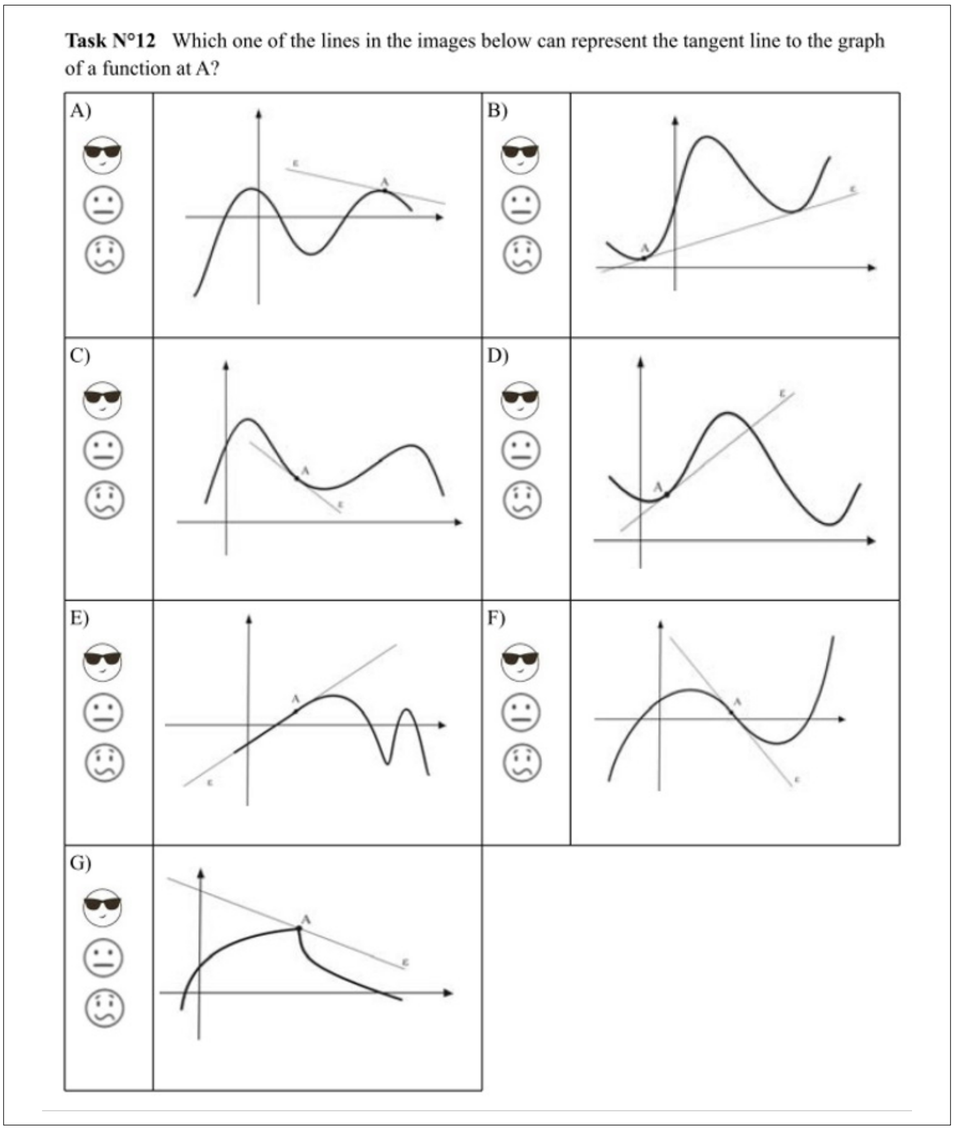

Figure 14
Task IV.3 – 12MO, 13VR.

Table 8 shows the great difference between the performances of the two classes in this task. The very first remark is that the students in 12MO class performed better than the others in every item. These students also showed a greater variety and richness in the answers to the task IV.3. If we focus separately on the answers of the students in the two classes, we see that the students in 13VR class performed better in the item A and C which, with respect to the situations depicted in the other items, entail situations that fit better with the *concept-image* of tangent line as a line which touches the function graph at a point. The number of right answer to items E and F is especially low. The former represents a situation which is not reconcilable with the *concept-image* of tangent line as a line which touches the function graph at a point; while the latter represents a situation which is not reconcilable with the *concept-image* of tangent line as a line which leaves the function graph in a half-plane. This last aspect indeed did not emerge explicitly from the students' answers to the task IV.3. The situation in 12MO class is different: the percentage of right answers is greater than 70% for every item. The items with the highest percentage of correct answers are F and C, while the percentage of correct answers to the item A is in the third place. We did not expect that, since the situation depicted in the item A can be considered a prototypical representation of the tangent line to a function graph at a point.

Item	Option	13VR (13)	12MO (18)
A	**Yes**	**69,23%**	**83,33%**
	No	23,08%	16,67%
B	**Yes**	**46,15%**	**72,22%**
	No	53,85%	27,78%
C	**Yes**	**61,54%**	**88,89%**
	No	30,77%	11,11%
D	**Yes**	**46,15%**	**72,22%**
	No	46,15%	27,78%
E	**Yes**	**23,08%**	**77,78%**
	No	69,23%	22,22%
F	**Yes**	**23,08%**	**100,00%**
	No	76,92%	0,00%
G	Yes	46,15%	22,22%
	No	**53,85%**	**77,78%**

Table 8
Percentages of answers - Task IV.3.

In general, comparing the tasks IV.2 and IV.3 we can see that the students who performed better in the task IV.3 are those students who mentioned explicitly, in IV.2, that the tangent line is locally indistinguishable from the function graph or that the slope of the tangent line is the derivative of the function at the point of tangency. Globally speaking only 8 students answered correctly to all the items of the task IV.3: 4 of them referred to the indistinguishability between tangent line and graph when answering the task IV.2, 3 referred to the derivative, and one did not answer.

6 Conclusions

In this paper we presented the preliminaries results of a pilot-study carried out in 2018. The aim of this study was to inquire the students' *concept-images* in NSA, and discuss, support or reject the idea that NSA is closer to students' intuition and spontaneous conceptualization. As said before, this idea is one of the reasons behind teaching proposals in both university and upper secondary level.

In this study, it emerges that in many contexts the *concept-images* developed by the students who attended a course in NSA are not mathematically adequate and efficient, at least not more than those developed by students who studied standard analysis and those documented in literature.
In fact many students not only gave wrong answers, but also sometimes their respective answers were not consistent with each other, as in tasks II.1 and II.2 (if compared), III.2, or for some options in task II.3. In these cases the students did not seem aware of the conflict in their answers, or they were not able to retrieve and use the correct definitions to solve the conflict.
Furthermore, the answers to tasks II.2 and I.1 suggest that the *concept-images* of the real numbers are also inadequate: one can question whether the introduction of hyperreals can be successful if the students do not have an adequate *concept-image* of reals.
The answers to task II.3 raise doubts on the supposed simplicity of calculation with hyperreals. In NSA, as in other mathematical contexts, performing calculations needs a careful conceptual control, otherwise it can lead to errors.
As suggested by the answers to task IV.1, the easier tools to calculate the limit of a function, starting from the equation given by NSA, does not translate into the construction of an efficient *concept-image*.
Regarding the *concept-image* of the tangent line to a graph at a point, we think that the possibility to access the idea that the tangent line and the function are indistinguishable has interesting teaching potential. We also note, that for many students in this study this idea was not so easy to retrieve or use.
Clearly the small scale and exploratory nature of our study do not allow for drawing too general and hasty conclusions. Nevertheless, we think that our preliminary results suggest caution before claiming that mathematical abstract notions – NSA notions in this case – have an intrinsic intuitive nature. We do not mean either to oppose the idea that it is possible to pursue an approach to teaching calculus through the introduction of NSA, or to claim that the proposals devised so far cannot have a certain potential with that respect. But further research is needed to possibly examine such potential and discuss the possible way to use them efficiently.

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References

Artigue, M. (1991). Epistémologie et didactique. *Recherches en Didactiques des Mathématiques, 10*(2/3), 241-285.

Biza, I. (2007). Is the tangent line tangible? Students' intuitive ideas about tangent lines. In D. Küchemann (Ed.), *Proceedings of the Conference of the British Society for Research into the Learning of Mathematics* (Vol. 27 pp. 6 - 11). London.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp.153-166). Kluwer: Dordrecht.

Di Nasso, M. (2003). I numeri infinitesimi e l'analisi nonstandard. *Archimede, 1*, 13-22.

Ely, R. (2010). Nonstandard student conceptions about infinitesimals. *Journal for Research in Mathematics Education, 41*(2), 117-146.

Harnik, V. (1986). Infinitesimals from Leibniz to Robinson. Time to bring them back to school. *The Mathematical Intelligencer, 8*(2), 41-63.

Keisler, J. (1976a). *Elementary Calculus: An Infinitesimal Approach*. Boston, Massachusetts: Prindle, Weber & Schmidt. Disponibile in https://www.math.wisc.edu/~keisler/keislercalc-12-23-18.pdf (consultato il 14.01.2019).

Keisler, J. (1976b). *Foundations of Infinitesimal Calculus*. Boston, Massachusetts: Prindle, Weber & Schmidt. Disponibile in https://www.math.wisc.edu/~keisler/foundations.pdf (consultato il 14.01.2019).

Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics, 48*(2-3), 137-174.

Machover, M. (1993). The place of nonstandard analysis in mathematics and in mathematics teaching. *The British journal for the philosophy of science, 44*(2), 205-212.

Marzorati, A. (2018). *Uno studio sull'insegnamento dell'analisi non standard nelle scuole secondarie di secondo grado*. Tesi di Laurea Magistrale, Università degli Studi di Pavia, Manoscritto non pubblicato.

O'Donovan, R., & Kimber, J. (1996). Nonstandard analysis at pre-university level: naive magnitude analysis. In N. Cutland, M. Di Nasso & D. Ross (Eds.), *Nonstandard Methods and Applications in Mathematics* (pp.235-248). Cambridge: Cambridge University Press.

Robinson, A. (2013). *Analisi non standard*. Roma: Aracne. (Titolo originale: *Non-standard Analysis* pubblicato nel 1965).

Sullivan, K. (1976). The Teaching of Elementary Calculus Using the Nonstandard Analysis Approach. *The American Mathematical Monthly, 83*(5), 370-375.

Tall, D. (1990). Inconsistencies in the Learning of Calculus and Analysis. *Focus, 12*, 49-63.

Tall, D. (1993). Students' Difficulties in Calculus. *Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7 1992* (pp. 13-28). Québec, Canada.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Vinner, S. (1983). Conflicts between definitions and intuitions: the case of the tangent. *Proceedings of the 6th PME Conference* (pp. 24-28). Antwerp, Belgium.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publ.

Autori/Mirko Maracci* and Ambra Marzorati
*Dipartimento di Matematica "F. Casorati", Università di Pavia, Italy
mirko.maracci@unipv.it, ambra.marzorati@gmail.com