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Misleading factors in graphical representations: a catalyst for recognizing the complexity of interpreting statistical graphs

Fattori fuorvianti nelle rappresentazioni grafiche: un catalizzatore per riconoscere la complessità dell'interpretazione dei grafici statistici

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Abstract / Literature has clearly shown how statistical graphs can be intentionally misleading: graphical representation of statistical information can be designed and realized with the intention of persuading and changing readers' value judgments on the basis of deceptive perception; more generally, the interpretation process can be affected by using misleading factors. Taking an educational perspective, this article assumes the hypothesis that it might be possible to exploit the effects produced by specific misleading factors for designing a didactical intervention focused on the interpretation of statistical graphs and aimed at bringing out these factors and with them an awareness of the effects they produce. The article reports on a first experimentation where the analysis of the data collected shows how appropriately designed activities can make prospective teachers aware of the effect of misleading factors, but also and more generally foster the development of their awareness about the complexity of the interpretation process.

Keywords: graphical representations; statistical graphs; lie factor; intuitive rules; prospective teachers.

Sunto / La letteratura ha mostrato chiaramente come i grafici statistici possano essere intenzionalmente fuorvianti: la rappresentazione grafica dell'informazione statistica può essere progettata e realizzata con l'intento di persuadere e modificare i giudizi di valore dei lettori sulla base di una percezione ingannevole; più in generale, il processo di interpretazione può essere influenzato dall'utilizzo di fattori fuorvianti. In una prospettiva educativa, si ipotizza che sia possibile sfruttare gli effetti prodotti da specifici fattori fuorvianti per progettare un intervento didattico incentrato sull'interpretazione dei grafici statistici e finalizzato a far emergere tali fattori e con essi la consapevolezza degli effetti da essi prodotti. Questo articolo presenta l'analisi dei dati raccolti in una prima sperimentazione e mostra come attività opportunamente progettate possano rendere i futuri insegnanti consapevoli dell'effetto dei fattori fuorvianti, ma anche e più in generale, favorire lo sviluppo della loro consapevolezza della complessità del processo di interpretazione.

Parole chiave: rappresentazioni grafiche; grafici statistici; *lie factor*; regole intuitive; futuri docenti.

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1 Introduction

To make people reflective citizens (Organisation for Economic Co-operation and Development [OECD], 2019), it is increasingly necessary to educate them in the interpretation of statistical graphs: it is not only a matter of being able to read numerical data, but also to recognize relationships between these data that are meaningful in relation to the context (Balchin & Coleman, 1966; Ben-Zvi et al., 2017). Graphs, diagrams, or other figures can be intentionally designed with the aim of persuading and changing readers' value judgments on the basis of deceptive perception. Therefore, it makes sense to analyze some features of graphs that can be misleading, as clarified in the classic contribution of Huff (1993), who speaks of "How to lie with statistics", and as taken up by Parks and Yeh (2021).
From a didactical point of view, it is important to prepare teachers to deal with this argument in their classroom, thus educating prospective teachers not only to recognize the problem of possible misleading effects in reading statistical graphs, but also, and more generally, to become aware of the complexity of interpreting statistical graphs as a didactical problem.
As discussed in this article, it is possible to identify two main educational aims:

- on the one hand, an immediate aim is to make the graph reader aware that misleading factors can affect the interpretation of statistical graphs;
- on the other hand, a more general aim is to educate in the reading of statistical graphs by making explicit the complexity of the interpretation process, in particular, by making explicit the two levels of interpretation, that of the data and that of the relationships between the data.

Based on the assumption that misleading factors can produce the effect of leading toward misleading interpretation of data, the hypothesis is that this factors, and the effects that are expected after their use, can be used in teacher education with the intention of fostering awareness about the process of interpretation of statistical graphs, in particular, about the complexity concerning the two levels of representation: that of data and that of their relationships. This hypothesis inspired the design of an intervention focused on interpreting statistical graphs: the choice of the tasks, specifically the choice of the representations proposed to be interpreted, aimed at making misleading effects emerge and with them readers' awareness of the different levels of representation. This paper reports on first results drawn from its implementation aimed at exploring its efficacy.

2 The issue in focus

Graphical representations, more or less directly related to statistical elaborations, are present in almost every area of everyday life (Gigerenzer et al., 1989): they are very common in politics and economics and often appear on TV, magazines, newspapers, or social media, and in many slide presentations created daily. These graphs are used for both data analysis and communication, and are often representations that students encounter in school curricula in disciplines such as science and social studies, and of course in mathematics and physics (Friel et al., 2001). Therefore, knowing how to correctly interpret these representations is an important skill to develop.
In the PISA framework (OECD, 2019), representation capability is meant as the ability of «selecting, interpreting, translating between, and using a variety of representations to capture a situation, interact with a problem, or to present one's work». Such representations include «graphs, tables, diagrams, pictures, equations, formulae and concrete materials» (OECD, 2019, p. 81). In mathematics, multiple

registers can be used to represent and communicate information. They can refer to qualitative as well as quantitative data. However, among the most common representations, graphics ones play a privileged role according to the shared assumption that they can support visualization. As Presmeg (2006) writes, «[…] visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics» (p. 206). Visualization often comes into play in the reasoning and problem solving processes (Arcavi, 2003; Stylianou & Silver, 2004), and this may be because it makes the data "perceptually easy" to interpret (Larkin & Simon, 1987). Indeed, given a sequence of numbers, the construction of a line joining them can produce a graphical representation that can add further information; for instance, the line can be interpreted as representing the trend of the sequence. Such a representation can be even more effective than that offered by a table: increasing, stability and decreasing of a particular phenomenon can be immediately grasped by looking at the inclination – upwards, flat, or downwards – of the line (Eshleman, 2002). In this respect, some authors claim that graphics are naturally interpreted (Zacks & Tversky, 1999), although many studies have pointed out that students have difficulty in interpreting some graphical representations correctly, for example histograms (Bruno & Espinel, 2009; Cooper, 2018; Cooper & Shore, 2010; Derouet & Parzysz, 2016; Lem et al., 2013; Watson & Moritz, 1998) or line graphs (Ali & Peebles, 2013; Kosslyn, 2006). In essence, what matters is how the observer interprets a graphical representation (Chabris & Kosslyn, 2005); for example, bar graphs in which the height of the bars must be evaluated are often interpreted correctly, while there are difficulties in comparing the relationships between the parts in a pie chart since it is a process that requires the comparison of "part/whole" relationships or between angles. These difficulties of interpretation often persist despite numerous educational interventions (Kaplan et al., 2014).

2.1 The interpretation of statistical graphs

There are two levels at which it is possible to interpret a given representation: that of data and that of relationships between data. Since there are different ways of representing data, each of these levels can highlight different aspects and, consequently, different ways of representing the relationships between data. For example, the two graphs in Figure 1 represent the same data (i.e., the numbers of bottles of wine sold in 2020 by the two salespersons A and B of the same company X) but in a different way.

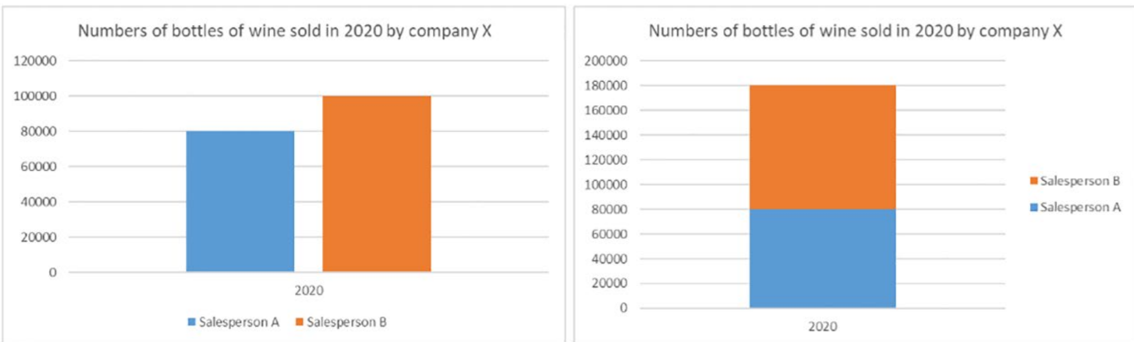


Figure 1. Graphs that represent the same information in different ways.

The graph on the left represents separately data relative to A and data relative to B. Whilst the graph on the right represents the two data combined into a single bar. As far as the relationship between the data is concerned, the two representations are not equivalent: on the one hand, while the difference in sales between the two salespersons is evident in the graph on the left, it cannot be perceived directly by looking at the graph on the right; on the other hand, the total of sales of the two salespersons is evident in the graph on the right, while it cannot be perceived directly in the graph on the left. Such interpretative variability inherent to the type of statistical graph used may explain students' difficulties both in dealing with graphs and in controlling what Cairo (2015) calls "misleading graphics", i.e. characteristics of graphs that can produce misleading interpretations. Henceforward, such characteristics are called *misleading factors*, a summary of which is given in sec. 3.

3 Conceptual framework

The design of the didactical intervention required to select interpretation tasks to be used consistently with the hypothesis formulated; thus, we considered the results reported in the literature on the possible effects produced by particular misleading factors. In addition, we took into account particular features of the interpretative process due to perceptual aspects, such as the focus of the observer on the *figure*, and not to what remains in the *ground*. The focus is on those misleading factors (*figure-ground* duality, distorting the system of reference), which can explain some of the difficulties in the interpretation of a statistical graph.

3.1 *Figure-ground* duality

In the interpretation of graphs some information about data can be evident, that is directly perceived, while others must be obtained through further elaboration. Thus, classical results about perceptual processes can be useful for identifying possible misleading factors, in particular the principles formulated by the Gestalt theory. Hartmann and Crosswhite (1966) emphasize that «all experience or mental life implies a differentiation of the sensory or perceptual field to which the organism can respond according to a *figure-ground* pattern» (p. 657). Therefore, there is a dualism between *figure* and *ground*, and this is a characteristic of every perceptual act. The *figure* is the characteristic of the situation to which the observer turns her attention, what is left is to be considered *ground* and has a secondary role. Thus, in an image the relationship between the parts and the whole may affect the perception process directing the observer's attention to one part (the *figure*) rather than another that consequently remains as the *ground*.

3.2 Distorting the system of reference

As far as a Cartesian or bar graph is concerned, there is a certain flexibility in the choice of the unit of measure on the axes: in case a non-monometric reference system is chosen, this can result into stretching or compressing one or both axes and make the appearance of the plotted data change. For example, in Figure 2, by compressing the y-axis in graph A and stretching the x-axis in graph B, the slope of the segment represented is perceptually altered. Thus, small differences in the variation of one of the variables may appear enlarged and, conversely, large differences may appear attenuated (Eshleman, 2002). This is a normal practice in the use of graphs (statistical and otherwise); for instance, this is the case of representations for which graphs are based on semilogarithmic or logarithmic reference systems, or representations of the globe or maps of transportation systems (such as subway maps); in these cases, the distances on the representation are not proportional to the actual distances, and proportionality is waived to highlight other aspects. However, such representations require the interpreter to have conceptual control over the representation process in order to ensure effective reading.

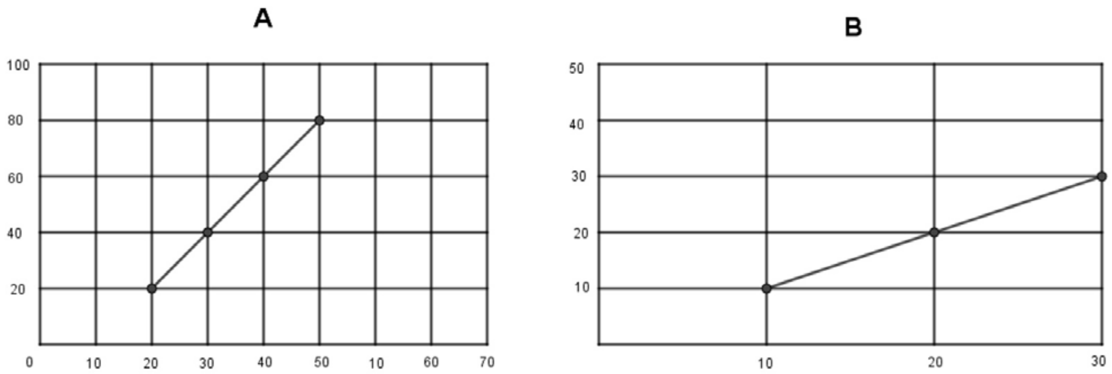


Figure 2. Stretching or compressing axes change the appearance of the plotted data.

As a matter of fact, for the author of the graph, there is the possibility of intentionally distorting the information in order to induce distorted reading. To describe this phenomenon that links a possible intentional distortion and its misleading effect, Tufte starts from the consideration that «a graphic does not distort if the visual representation of the data is consistent with the numerical representation» (Tufte, 1983/2001, p. 55), i.e. whether there is direct proportionality between the data representation, physically measured on the graph, and the data represented. In this regard, he defines *lie factor* the ratio between the size of the effect shown in the graph (that is the relative difference between the representations) and the size of the effect shown in the data (that is the relative difference between the data). According to the previous description of the intrinsic complexity of statistical representation, this definition of *lie factor* is interesting because it is particularly significant with respect to the distinction between the level of interpretation of data and the level of interpretation of the relationship between data. As it is mathematically defined, the *lie factor* is an index that measures any "distortion" of the graph: when the *lie factor* is equal to 1, there is no real distortion; when the *lie factor* is greater (less) than 1, then the size of effect shown in the graph is greater (less) than the size of effect shown in data, which leads the interpretation to an overestimation (underestimation) of the relation between data. **Figure 3** below shows two examples of bar graphs made from the same data.

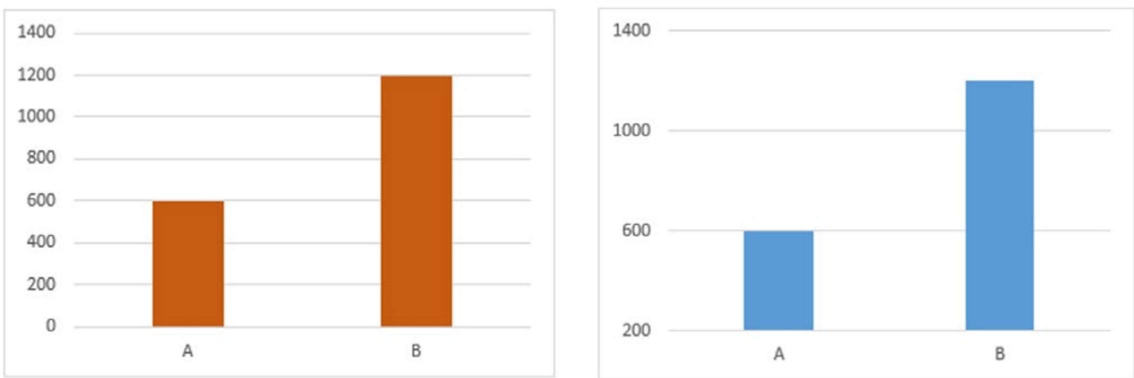


Figure 3. Two examples of bar graphs with different *lie factor* made from the same data.

In the graph on the left, the *lie factor* is equal to 1 because the heights of the bars are proportional to the values reported on the vertical axis. In the right graph, however, there is no proportionality. Indeed, bar "B" has a height that is more than twice that of bar "A", but this does not agree with the values on the vertical axis. In the right graph, therefore, the *lie factor* is greater than 1 and the ob-

server is led to overestimate the data. Tufte (1983/2001) reports several examples of severe *lie factor*. In some cases, the lack of direct proportionality between the data representation, physically measured on the graph, and the data represented can be an intentional misleading factor that produces misleading effects, as in the graph on the right in Figure 3.
In this paper, we intend to bring out such effects of distortion in order to lead students and teachers to reflect on these issues, and in general on the process of interpreting graphs; this is made clear in the following section.

4 Research hypothesis and research question

The conceptual framework described above has guided the research project and inspired the design of a didactical intervention whose aim is to make participants aware of the complexity of interpreting statistical graphs and graphs in general.
The hypothesis concerning the role played by misleading factors in becoming aware of the complexity of the interpretation process guided the design: misleading factors were intentionally introduced into interpretation tasks with the expectation not only that the related effects would arise, but also that the awareness of these effects would trigger reflection on the interpretation process and the complexity of that process. Thus, the research hypothesis and the research question have been formulated as follows:

- *Research hypothesis*: the introduction of misleading factors in graphs can lead the task solver to become aware of the complexity of the interpreting process; in any graph the representation of the data determines also how the relationships between them are represented.
- *Research question*: is there any evidence that the designed didactical intervention can promote awareness of the complexity of the interpretative process? In particular, is there any evidence of the influence of misleading factors on the triggering of such awareness? More specifically, is there any evidence of the influence of misleading factors on the triggering of such awareness?

Considering the importance of educating in the interpretation of graphs, and statistical graphs in particular, the target group of the intervention are students in the final year of the master's degree programme in mathematics (hereafter referred to as "participants"), considered as potential prospective mathematics teachers.

5 Method

In the following, we describe the design of the didactical intervention, the participants and the experimental setting, data collection and the criteria of data analysis.

5.1 The design of the didactical intervention

The research hypothesis was explored on various types of graphs in order to broaden the case study as much as possible. For this reason, the design of the didactical intervention was centered on tasks asking to interpret different types of statistical graphs: line graphs and bar graphs. For each of the two types, three different graphs were produced representing the same set of data but differing in the way they represented the data and consequently the relationships between them. Differences are due to the introduction of particular misleading factors: for instance, the *y*-axis is omitted, it is zoomed, the values reported do not start from the origin of axes, proportionality is not respected or the size of the bars is stretched.

5.1.1 Design of graphs used in the didactical intervention

The following misleading factors were taken into account when creating the graphs.

– M1. In accordance with the principles emphasized by Gestalt and the studies of Hartmann and Crosswhite (1966), in a statistical graph, the relationship between the parts and the whole may affect the perception process directing the attention of the observer and consequently the focus on what is to be considered the *figure* and leaving what remains as the *ground*;
– M2. In accordance with the studies of Tufte (1983/2001), the presence of a *lie factor* greater (less) than 1 may affect the interpretation of a statistical graph leading to overestimate (underestimate) the relationship between data.

The complete collection of the created graphs is reported in Table 1, which is organized to display in the same column the three graphs of the same type.

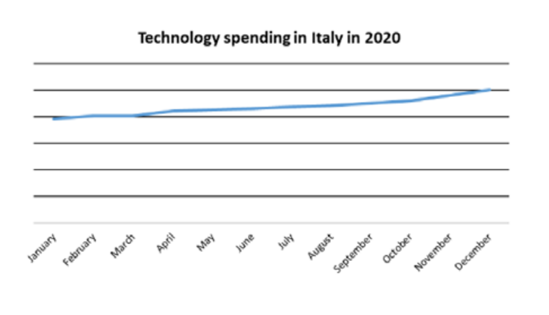
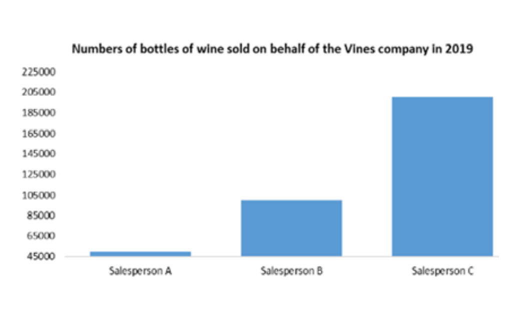
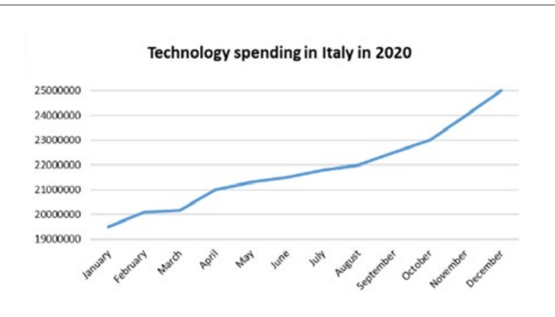
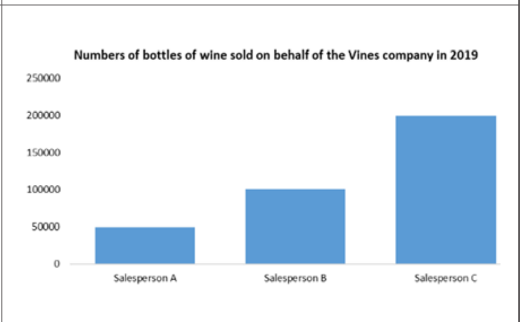
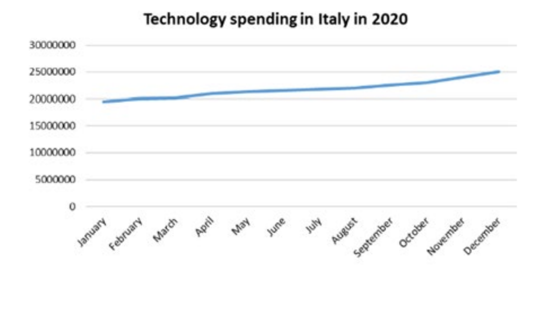
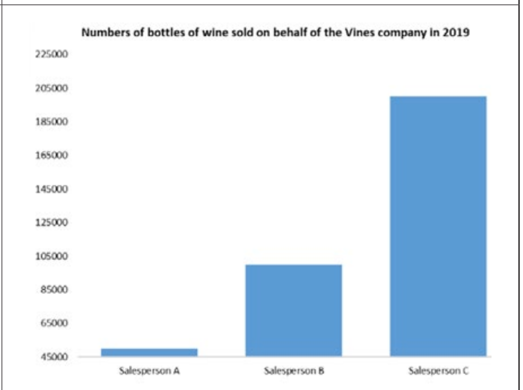
	1	2
A		
B		
C		

Table 1. The three versions of statistical graphs according to the two types.

In the following, we will denote the graphs in Table 1 using the row and column reference. For example, graph C1 is the graph in the lower left corner of Table 1.
As far as the line graphs are concerned, the data displayed in the three graphs are the same and report the "Technology spending in Italy in 2020", but their appearance differs because of some misleading factors. The trend of the curve, shown by a slowly increasing line, is the same in graphs A1 and C1, but in graph A1 the *y*-axis is omitted. In accordance with the misleading factor M1, the absence of the values on the *y*-axis may be unnoticed by the observer (*ground*). In the case of graph B1, the *y*-axis does not start at 0 (misleading factor M2). The result is a graph that perceptually increases rapidly, and this might lead to the interpretation of a great increase in spending.
As far as the bar graphs are concerned, the data displayed in the three graphs are the same and report the "Numbers of bottles of wine sold for company Vines in 2019" by three salespersons (Salesperson A, Salesperson B, and Salesperson C). In graph A2, the values on the *y*-axis do not start from 0 (a detail that might not be noticed by the observer and go into *ground*, in accordance with the misleading factor M1) with the aim of perceptually belittling the work of Salesperson A, whose column has a height, and therefore an extension, of almost zero. Graph C2 is obtained from graph A2 by lengthening the *y*-axis, thus generating a graph with *lie factor* greater than 1 which can lead the observer to overestimate the data, in accordance with the misleading factor M2, and to perceive a very strong difference between Salesperson C and the others. Graph 2B shows the information "honestly", i.e. the values on the *y*-axis start from 0, but are less detailed.

5.1.2 The structure of the didactical intervention

The didactical intervention consisted in three phases with different interpretation tasks, inspired by the research hypothesis about the role that misleading factors may play in triggering students' awareness of the complexity of the interpretation process.
Phase 1 aimed to make the solver focus on interpreting the various graphs described above. The task consisted of presenting the six graphs in Table 1, one at a time, so that graphs of different types were shown consecutively: first the graphs in the first row (A1 and A2), then those in the second row (B1 and B2), and finally those in the third row (C1 and C2). For each graph, participants were asked to answer the following requests: *Look at the graph. What could you infer? What information is it trying to communicate? Justify your answer. Moreover, attribute a meaningful title to the graph.*
According to the main hypothesis, such questions were intended to guide towards the interpretation of the different graphs in terms of the relationship between the data. In this respect, the request of a meaningful title is a key element: as a matter of fact, at this early stage of the activity, the expected answers to this request can reveal the misleading effect of the misleading factors.
Phase 2 aimed to make the solver focus on the difference between representation of data and representation of relationships between data; according to the main hypothesis, the activity was based on the comparison of different graphs: the expectation was that the effect produced by specific misleading factors might promote solvers' awareness of discrepancies and inconsistencies, thus leading them to become aware of the complexity of the interpretation process. In phase 2, all graphs of the same type were presented simultaneously: first the line graphs (A1, B1, and C1), then the bar graphs (A2, B2, and C2), i.e. the three graphs that differ in terms of the effects of misleading factors were presented together. The task was formulated as follows: *Look at the graphs. Do you think they say the same thing? Do they represent the same information? Are they trying to communicate the same information? Justify your answers and report your reflections.*
The explicit request of comparison was expected to lead the solver to analyze the three cases looking for similarities and differences: the effect of misleading factors might emerge leading the solver to recognize the difference between representations of data and representations of relationships between data.
Phase 3 aimed to foster participants' reflection on the whole activity carried out in relation to their

possible future role as teachers. For this purpose, each participant was asked: *Report your reflections on the activity. As a prospective teacher, dwell on educational aspects that you think are important.*

5.1.3 Participants and organization of the didactical intervention

The participants in the experiment were 19 students of the second year of the Master's degree in mathematics (therefore already in possession of a Bachelor's degree in mathematics) who were attending the Mathematics education course at the University of Campania "L. Vanvitelli". The participants carried out the activity during the course lectures and agreed to the data collection.
The activity was carried out in two consecutive lessons. In the first lesson, lasting 2 hours, phase 1 took place. In the second lesson, lasting 3 hours, both phase 2 (about 2 hours) and phase 3 (about 1 hour) were experimented. Participants were given the possibility to think carefully about each question and formulate their answers unhurriedly. The teacher did not intervene during the activity.
All phases were implemented on Moodle platform using the "Questionnaire" module. Each phase was implemented with a different questionnaire. By means of Moodle's "Terms of access" feature, the questionnaires were constrained to each other, meaning that participants only had access to subsequent questionnaires after completing and submitting the previous ones. In this way, students addressed all phases sequentially, according to the design. In addition, again in accordance with the design, by setting the "Visibility of submitted questionnaires" option to "never", participants were unable to view their own answers submitted in previous steps.
All data were collected through the Moodle platform.

6 Data analysis and results

A qualitative analysis of the participants' responses and comments was carried out, searching in collected data for possible interpretation of the representations showing evidence of the expected effect of misleading factors. Because of the role that any phase of the sequence was expected to play, the report of the analysis has been structured according to the three phases and their specific aims.
Participants were denoted by the labels P1, P2, …, P19.

6.1 Phase 1

In this phase, specific attention was paid to the title that participants were asked to attribute to the graphs. According to the misleading factor M1, the initial focus of most participants was on specific parts of the graphs, neglecting others.

Line graphs. Regarding the line graphs, most participants seem to have been affected by the misleading factor M1. As expected, for graph B1, the majority (fifteen participants) overestimated the inclination of the line and referred to a significant increase in spending. For graph A1, all participants, however, referred to an increase in spending. Specifically, five participants referred to «steady» growth, two to «almost linear» growth, and one to «progressive» growth. However, sixteen participants pointed to «slow growth». Nine participants underestimated the information represented by graph C1, and referred to «slight increase», «small increase» or «regular increase» in spending. Only three participants mentioned a «high increase».
Even when more representation details were perceived, the effect of misleading factors M1 and M2 persisted.

In the case of graph B1, some participants noted the presence of the values on the *y*-axis, omitted instead in graph A1. Here are some examples:

- P2: «From that graph [B1] it is possible to infer an increase in spending in Italy. Unlike the previous graph [A1], this increase is supported by values».
- P15: «In this case [B1] the unit of measure shown on the *y*-axis makes the quantitative information readable».

Other participants referred to more detailed information, for instance:

- P7: «A less regular trend can be inferred in this case [B1], since the information is more detailed and consequently the graph is more accurate».
- P15: «In this graph [B1] we can see in more detail the expenditure that occurred in 2020».

Some participants explicitly referred to numerical values, as:

- P18: «It can be deduced that expenditures have increased, increasing from 19'000'000 in January 2020 to 25'000'000 in December, holding steady between February and March».

Generally speaking, the effects of misleading factors were evident as shown in the interpretations and especially the titles given by P1 and P4 (Table 2).

	Graph A1	Graph B1	Graph C1
P1	«It can be inferred that in the January-March period there is a constant spending, while in the April-December period there is an increase in spending. Title: Increased spending».	«There is a substantial increase in spending at the end of the year compared to January. Title: Increasing in spending».	«By the end of the year there is a slight increase in spending. Title: Slight increase in technology spending».
P4	«The graph shows that over the course of a year, spending increased progressively. Title: Money spent on technology».	«During the year 2020, spending on technology increased almost exponentially. Title: Soaring increase in technology spending».	«The spending in technology has increased steadily during 2020 in Italy, from € 20'000'000 in the previous year, to as much as € 25'000'000. Title: Technology spending continues to increase in Italy».

Table 2. Interpretation of the line graphs given by P1 and P4.

Nevertheless, already in phase 1, as the different graphs follow one another, some explicit remarks on possible misleading interpretation can be observed. A few participants began to compare the graphs of the same type displayed at different times and seem to have recognized a difference between the representations of data and the representations of the relationships between data.
For example, with regard to the line graphs, P9 noted that the last two graphs (respectively C1 and B1) represented the same information but in different ways. P9 wrote:

> «I have noticed that the values actually do not differ from those in the previous graph which showed the same information. However, the line does not have a very visible increase and therefore suggests a slight increase in spending. I give it the title "Slight increase in technology spending"».

P9 noted that the values in graph C1 are the same as in graph B1, and that, for the same data represented, the representation of the relationships between the data may differ: in graph C1, «the line does not have a very visible increase», so graph C1 «suggests a slight increase in spending»; specifically, P9 confirmed the difference in the interpretation of the relationship between data, formulating the title «Slight increase in technology spending».

Bar graphs. Similarly, the effect of misleading factors emerged in the case of bar graphs. With regard to graph A2 and B2, most participants seem to have been influenced by misleading factor M1 and focused on the heights of the columns and, in particular, on the tallest column (*figure*). Only four participants seem to have focused on the values on the *y*-axis, and no participant noticed that the *y*-axis did not start at zero (*ground*). As a result of misleading factor M2, twelve participants, referring to graph C2, stated that Salesperson C was by far «the best salesperson». The participants seem to have focused on the size of the images: they considered the seller to be better the higher the column of the graph referring to him, regardless of the values on the *y*-axis. In these cases, too, the same expressions were used, «significant increase» and «slight increase».
Going into the details of the information provided by individual participants, also with regard to the bar graphs, P9 wrote:

> «From this graph [C2], I always notice a greater sale of Salesperson C compared to the others. Actually, reading the scale on the left the values are not different from those of the previous graphs. However, it suggests that Salesperson A sells next to nothing, while Salesperson C is a success. I titled it "Supremacy of Salesperson C"».

P9 claimed that graph C2 represents the same data as the graphs displayed earlier («reading the scale on the left the values are not distinct from previous graphs»). However, the biased way the relationship between data is represented in graph C2 takes over on that displayed in the other graphs of the same type. P9 wrote that graph C2 «suggests that Salesperson A sells almost nothing, while Salesperson C is a success» and finally, P9 assigned the graph the title «Supremacy of Salesperson C».
Also P8 showed to be aware of the differences in representing the data, but the effect of misleading factor M2 emerged; actually, P8 wrote:

> «The *y*-axis compared to the previous graph is much more spaced out, so the answer becomes much more evident: Salesperson C sold much more».

To some extent, some of the participants already seemed to discriminate the difference between the representation of data and the representation of the relationships between data, but this remained rare and limited.

6.2 Phase 2

In phase 2, when the comparison between graphs was explicitly requested, the analysis focused on evidence of different interpretations of graphs of the same type showing the emergence of awareness that differences in the interpretation could be induced by specific choices in the representations. The analysis was primarily focused on the emergence of commentaries revealing a shift in focus from data representation to different possible representations of their mutual relationships; in particular, we were interested to observe to what extent the effects of a misleading factor contribute to the development of interpretation awareness.
As expected, in phase 2, the explicit request of observing and comparing the graphs of the same type and displayed simultaneously led the participants to notice details not previously observed, though not all of them seemed to have reached the awareness of the whole complexity of the interpretation process and be able to fully describe it.

The analysis showed evidence of participants' awareness of differences between the graphs of the same type. Almost all the solvers realized of differences but only a part of the participants (nine) seemed to be aware of the fact that such differences affect the interpretation of the relationship between data and explicitly remarked that although the graphs represent the same data, they actually represent different relationships among data. Attention focused on specific parts of the graphs, e.g. the figures, or the labels, «the presence of the unit of measure...» or «the height of the bars...». This shift of focus was accompanied by different interpretations introduced by expressions such as «the graph suggests that...» or «the graph wants to communicate that...», which witness to an increased level of awareness about the intentionality for graphs to communicate specific information and may correspond to a reached awareness about the impact that specific choices for representing data can have on the interpretation of the relationship between data.

Line graphs. With regard to the line graphs, P4 wrote:

> «Obviously, the information communicated is the same, what changes is how, and, above all, what is being emphasized: in the first, only the annual increase is shown, without specifying the sum; in the second, the intention is to show, in particular, the monthly variation in spending; in the third, to show the annual variation, showing numerically the difference from January to December».

P4 claimed that «the information communicated is the same» but seems to have reflected on the fact that the graphs want to emphasize different aspects; however, it is unclear how much "emphasis" could become "misleading". A deeper analysis was carried out by P3, P12 and P9, who, respectively, used expressions referring to the intentionality of the representation process and consequently to how open interpretation can be. Expressions such as «wants to highlight», «it seems that the increase is less than in the second graph» and «meant to communicate/suggest» express the intentionality of the graphs to show specific relationships between data:

- P3: «[...] Graph 2 [B1] instead wants to highlight how technology spending just increases month by month».
- P12: «[...] Looking at the last two [B1 and C1], the graphs are meant to communicate the same information, but in the last graph it seems that the increase is less than in the second one, because in the second one, spending is shown in more detail».

P3 and P12 seemed to have recognized a difference between representation of data and relationships between them, but they did not make it explicit. On the contrary, P9 wrote:

> «[...] the second and the third represent the same information but do not want to communicate the same information. Indeed, the second one is meant to suggest an increase in spending, while the third is meant to suggest a slight increase in spending».

P9 distinguished between the data represented by the graphs («the second and the third represent the same information») and what they intentionally wanted not to communicate («but do not want to communicate the same information»). P9 used the same words, «the same information», once to refer to the data, and then to refer to the relationships between the data (increase of spending). P9 seemed not to be aware of the inconsistency of this use or maybe he tried to express the relation between representing and communicating as well as its complexity.

Bar graphs. P9 gave a quite similar response in the case of bar graphs:

> «Again, the graphs all represent the same information but communicate different information. The third graph [C2] suggests that only Salesperson C sells, Salesperson B sells little, and Salesperson A does not sell at all. The first graph [A2] conveys a gap between Salesperson C and the absolute others. The second graph [B2] instead shows a more homogeneous sale of products».

P9 used the expressions «the graphs all represent the same information but communicate different information», however specified: «the third graph suggests…» and «the first graph conveys…» to indicate that the data represented are the same, but the ways they are represented in the different graph suggested different interpretations. Actually, generally speaking, in the case of bar graphs, similar responses can be found; for instance, P3 emphasized the difference in the possible interpretation induced by the intention of communicating different information and talked explicitly about distortion. P3 wrote:

> «They represent the same information, but they want to communicate different information. The wine of Salesperson C has been bought twice as many times as the wine of Salesperson B which has been bought twice as many times as the wine of Salesperson A. Graph 1 [A2] and graph 3 [C2] want to distort this information. In these graphs there is a tendency to minimize the bottles of wine of Salesperson A and Salesperson B».

The use of the expression «Graph 1 and graph 3 want to distort» shows that P3 recognized the effect of misleading factors. P4 also recognized the effect of misleading factors. Indeed, he noted the stretching of the *y*-axis in the graph C2 (misleading factor M2) and the effect that this stretching perceptually causes, that is a «less clear-cut gap» in sales. P4 wrote:

> «Here, too, the information is the same: the way the result is presented changes; the effect obtained by decreasing the scanning of the bottles sold, and therefore the height of the bar graphs, is a less clear-cut gap in the second compared to the third».

This visual effect was also observed by P5, who used the expression «visually the impact is different». He wrote:

> «At a quantitative level they report the same information, but visually the impact is different because from the second to the third one, the important gap on sales of C compared to A and B is highlighted».

For P8, it was necessary to go beyond the «visual effect» to fully understand what the graphs were trying to communicate, i.e. the relationships between data:

> «When you read them, the visual impact is certainly important, but you need to focus more on the data in a graph. The intervals of the third graph have a greater amplitude and, as a result, make it clearer what we want to convey».

P8 noted the misleading factors («the intervals of the third graph have a greater amplitude») and their effect which results in a different interpretation of the graph.

6.3 Phase 3

The comments provided in phase 3 were analyzed with the aim of providing an answer to the research question concerning the awareness about the complexity of the interpretation process for statistical graphs, and how far such awareness is to be related to the presence and the effects of misleading factors.
From the analysis of participants' answers in phase 3, we get evidence that most prospective teachers increased their awareness of the interpretation process and of its complexity. Most of the participants (eleven) emphasized the importance of focusing on the details when interpreting the graphs. For example, P16 wrote:

> «What is really important at the end of this activity is dwelling on the data at hand and analyzing it in detail».

P11 emphasized the importance of looking at the «trend of the graph» but also the «scale used». He wrote:

> «I think this activity is useful for students to learn to look at a graph in its entirety, i.e. values entered, scale used, trend of the graph, and how to derive useful information from it».

P5 also spoke of details, but also of an awareness of the errors made, which gradually matured during the activity:

> «During these different questionnaires, I gradually became aware of the evaluation errors I had made in the previous steps, which emerged precisely from the comparison of similar situations and with greater reflection on the details».

Recalling the effects of misleading factors experienced in previous phases, P8 referred to the fact that sometimes graphs can be intentionally designed to induce a specific interpretation. Thus, he stressed the importance of being able to read graphs to understand what information they want to communicate and detect possible misleading intentions. Indeed, P8 wrote:

> «Knowing how to read a graph doesn't make you a slave to what others want you to understand, but you can have your own clear and immediate idea».

Similarly, P3 and P9 explicitly referred to deception and lies. P3 wrote: «Graphs need to be well interpreted because they can be deceptive». However, P9 pointed out both the need of being attentive but also the need of a specific ability for interpreting a graph; he wrote:

> «This activity suggests me that the information one wants to convey is easily manipulated if one looks at it in a distracted way or if one doesn't know how to read graphs in a certain way. I think it is essential to be able to read a graph and not be "fooled" by what they want to convey».

Beyond a generical focus on the importance and the complexity of acquiring a specific competence for interpreting statistical representation, some of the participants attempted to explain what such competence might consist of. For instance, two participants, P1 and P17, seemed to be aware of the need of considering how data are represented and how the representation modality can affect the interpretation of the relationships between them.

- P1: «We need to dwell on the details because even equal graphs appeared different».
- P17: «[...] for example, the graphs on wine sales at first, because of how they were represented, might have seemed different, but actually evaluating the data we can see that they represent the same results».

In both cases, the role of the effects produced by the misleading factors appears («even equal graphs appeared different», «graphs… might have seemed different, but actually…»); P1 and P17 seem to have achieved awareness of the interpretation process reflecting on their experience and specifically on the effect that different ways of representing a set of data can have on the interpretation of their relationship: how the number of sold bottled is represented, and how this can change the way of representing the sale rate of the wine salespersons.
Participants' reflections also considered the didactical relevance of dealing with these issues in class with students. For example, P10 wrote:

> «I think this activity would be useful in schools to show how the same diagrams can be misinterpreted in different ways. It is important because by working alone and giving an autonomous answer, it is the student him/herself who realizes whether or not he/she is wrong when looking at the next diagram. Making mistakes is a great way to learn».

It seems that P10 emphasized that the activity allows participants to autonomously realize their mistakes, and that this is a necessary step in learning. Very interesting is the answer of P15, which definitely supports our hypothesis about the efficacy of the designed activity:

> «A superficial observation of the graphs could lead to inferring wrong information. It is important to develop skills of observation and reflection, as well as technical tools, so that the first information the eye catches is not the final one. Being able to quantify data is the only way for our perception not to be distorted. A didactical aspect that I think is important is certainly the understanding that not everything that is presented to us is so immediate; the tools acquired during our studies can be challenging, but are fundamental so that our eyes are, in fact, less deceivable».

7 Discussion and conclusion

In interpreting a statistical graph, inferences are made about the relationships between the data represented, and these may suggest conclusions that may turn out to be quite different. In particular, it may be that specific representation modes induce distorted conclusions (Pastore et al., 2017; Weissgerber et al., 2015). This is the case with those which, in this article, have been called misleading factors.
The research presented in this paper stems from the hypothesis that misleading factors can be didactically used to trigger awareness of the complexity of the interpretation process that involves at the same time the interpretation of data and the interpretation of the relationships among them. Starting from this hypothesis, we designed a three-phase didactical intervention related to the didactical problem of interpreting statistical graphs. This problem arises at two levels: realizing that graphs can be affected by misleading factors; promoting awareness of the complexity of the interpretation process of a statistical graph. The didactical intervention has been experimented with students in the last

year of the Master's degree in mathematics. The analysis of the data collected during this experiment seems to support the reasonableness of the research hypothesis. The analysis showed evidence of the emergence of the effect of misleading factors, but also of an increasing awareness of the complexity of the interpretation process; specifically, participants seemed to reach awareness of the intentionality of the representation process and consequently how open interpretation can be particularly with regard to the relationship between the data represented. Thus, the analysis presented above can provide an initial verification of the research hypothesis. Though not all participants went into depth in making the relationship between data explicit, many of them compared graphs and focused both on the data and on their relationship, seeking elements that could make it explicit. Moreover, as witnessed in the comments provided in phase 3, the didactical intervention allowed participants to experience moments in which they took on a dual role, first as students, grappling with reading and interpreting the graphs, and then as future teachers, reflecting on the educational aspects related to the activity. As expected, phase 3 provided an opportunity for participants to reflect, not as students but as prospective teachers, on the crucial role played by specific aspects of graphical representation in the statistical field and the educational importance of addressing these issues in the classroom. As clarified above, the contribution of this research does not relate directly to statistical concepts since it addresses the educational problem of graph interpretation. However, it addresses a key element of statistical knowledge, namely representations of data and the inference processes that arise from those representations. From this point of view, this research contributes to the development of teachers' knowledge (Shulman, 1986) in statistics (Groth, 2007).
The results of this study also seem to show that the research hypothesis can be translated into a design principle for didactical interventions: using graphs constructed on specific misleading factors to make students aware of the complexity of the interpretation process. Obviously, activities similar to the one described, analyzed and discussed in this paper are only a prerequisite for later classroom discussion with students on aspects of interpreting statistical graphs.
There is another interesting aspect that emerges from the data analysis and concerns a more general issue. By looking at a graph and interpreting some of its features, it is possible to directly infer not only properties of what is represented, but also other properties related to the context, that is the situation from which the data are extracted. This inference can be made automatically without necessarily making explicit the relationship to the data itself. From the data analysis emerged, for example with reference to the bar graph C2, how twelve participants, seeing that the Salesperson C's bar was the highest of all, automatically inferred that C was the best salesperson overall. In our view, the described phenomenon could be interpreted in accordance to intuitive rule theory (Fischbein, 2005; Stavy et al., 2006; Tirosh & Stavy, 1999) and, in particular, to the intuitive rule "more A – more B", according to which the A1 > A2 comparison, in reference to a quantity A, extends to the B1 > B2 comparison, in reference to another quantity B that is considered related to A. For example, if an observer sees objects of the same shape, of which the first is heavier than the second, thrown from the same height, he/she tends to conclude that the first will reach the ground before the second. Or, if an observer sees two angles (opposite the vertex) with the sides of the first represented longer than the second, he/she tends to conclude that the first angle is greater than the second. This intuitive rule "more A – more B" might explain some of the misinterpretations of statistical graphs. In phase 1, most participants behaved according to the intuitive rule "more A – more B", regardless of whether or not the values on the *y*-axis were present in the graphs. Indeed, most of the participants considered:

- excessive the expenses which were represented where the line had a greater slope;
- the best salesperson, the one whose sales were represented by the highest column.

In this regard, Tversky (1997) points out that people, when reading a graph, tend to give more importance to the vertical dimension than the horizontal one and to see "high" as "more" or "better". The

vertical dimension, in fact, is more significant in interaction with the world (e.g. gravity acts vertically, people grow vertically). This could explain why certain representations seem to be more difficult for students to interpret than others. From this perspective, this study can contribute to the international research on intuitive rules (Fischbein, 1997; Stavy et al., 2006; Tirosh & Stavy, 1999). Indeed, the data analyzed in this paper seem to show that such rules can also involve aspects related to statistics and the reading and interpretation of graphs.
In this respect, this work opens a research direction on perceptual aspects related to the reading of statistical graphs: gaining a deeper understanding of how and why reading a graph can be misleading, could provide key elements for designing educational interventions to develop teachers' didactical sensibility about the development of students' representation capability. However, this study involved only a limited number of prospective teachers. From this perspective, further research involving more prospective and in-service teachers is needed to analyze the process of interpreting a graphical representation. It would also be interesting to broaden the investigation to other mathematics domains, or to mathematical modeling, for which the interpretation of specific graphs presents similar complexities.

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