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Secondary teachers' geometrical figure apprehension: their ability to construct geometrical proof and to predict students' difficulties

La comprensione delle figure geometriche da parte degli insegnanti di scuola secondaria: la loro capacità di costruire dimostrazioni geometriche e di prevedere le difficoltà degli studenti

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Abstract / The present study investigates in-service secondary teachers' geometrical figure apprehension in relation to their ability to construct geometrical proofs and to predict didactically their students' difficulties and mistakes. The theoretical framework of analysis is based on Duval's geometrical figure apprehension which was taught to the teachers as part of an in-service training course in didactics of mathematics, offered by one of the researchers. As part of the course final assessment, a written test consisting of various geometrical tasks was constructed and administered to the sample. Participants' answers in both solving and interpreting difficulties related to the tasks were analyzed. The results of the study indicate that theories and concepts of didactics of geometry can shed light to various facets of teaching and learning of geometry. A teacher who presents a correct solution at a task does not necessarily identify or understand the possible difficulties faced by students. Discussion concentrates on teaching implications about geometry and geometrical proof.

Keywords: geometry; geometrical figure apprehension; geometrical proof; difficulties; secondary school teacher training.

Sunto / Questo studio analizza la comprensione delle figure geometriche da parte dei docenti di scuola secondaria in servizio, in relazione alla loro capacità di costruire dimostrazioni geometriche e di prevedere didatticamente difficoltà ed errori dei loro studenti. Il quadro teorico di analisi è basato sulla teoria della comprensione delle figure geometriche elaborata da Duval, presentata ai docenti nell'ambito di un corso di formazione continua in didattica della matematica. Come parte della valutazione finale del corso, è stato costruito e somministrato un test scritto con varie attività geometriche. Sono state analizzate le risposte dei partecipanti in termini di risoluzione e di interpretazione delle difficoltà. I risultati indicano che le teorie e i concetti di didattica della geometria possono far luce su vari aspetti dell'insegnamento-apprendimento della geometria. Un docente che propone una soluzione corretta non necessariamente identifica/comprende le possibili difficoltà degli studenti. La discussione si concentra sulle implicazioni didattiche relative alla geometria e alla dimostrazione geometrica.

Parole chiave: geometria; comprensione delle figure geometriche; dimostrazione geometrica; difficoltà; formazione degli insegnanti di scuola secondaria.

1 Introduction

Geometry, whose content area is shapes and objects, has an essential place in human life in science, art, architecture, engineering (Yavuz et al., 2016). At the same time, the teaching and learning of geometry is a privileged domain of research among researchers in mathematics education and psychology. «Geometry instruction is an important yet often overlooked subject in current education research and practice» (Bergstrom & Zhang, 2016, p. 134). There is also growing consensus that geometry is a domain that causes difficulties to students, independently of their age and culture.

Since the early 1970s, numerous studies carried out in several countries had evaluated teachers' mathematics subject knowledge based upon the common belief that the greater the subject knowledge teachers have, the better they teach. In the case of geometry, mathematical subject knowledge among teachers appears uneven, with a lot of gaps, especially among primary school teachers (Leikin & Levav-Waynberg, 2007). Yet, the link between limitations in subject knowledge and the quality of teaching is clear. There are different conceptual frameworks describing mathematical knowledge that is needed for teaching (Manizade & Mantinovic, 2018). Researchers generally agree that pedagogical knowledge, as introduced by Shulman, connects knowledge of mathematical content with pedagogy. Shulman (1986) discusses the necessity to examine teachers' knowledge and ability to teach in relation to the respective content knowledge. Teachers' mathematical content knowledge contributes significantly to students' achievement (Chinnappan et al., 2018). The knowledge a teacher brings to the teaching-learning context is fundamental to the quality of student learning. According to Ball et al. (2008), the greater the range and depth of content knowledge, the greater the teacher's ability to flexibly extent this knowledge to his/her pedagogical or mathematical knowledge for teaching.

Students' performance in geometry has been related scientifically and didactically with the ability to understand and construct geometrical proofs. Cirillo (2018) argues that although there have been ongoing calls to improve the treatment of reasoning and proof in school mathematics, success in teaching proof has remained elusive. The findings of international research confirm that the teaching of geometrical proof in secondary school is not an easy task (Fujita & Jones, 2014). As Fujita et al. (2010) claim a key challenge in mathematics education in respect to geometry and geometrical proof is to devise ways of enabling students to successfully move between the practical and theoretical domains of mathematics. Successful teaching of geometry depends on teachers' knowledge of geometry and how to teach it effectively (Jones, 2000).

The present study aims to contribute to international efforts regarding the improvement of the teaching of geometrical proof through the lens of teachers' geometrical figure apprehension, and more specifically Duval's model, as well as their ability to teach effectively. A part of the teaching process which was examined through the present study has to do with teachers' ability to predict, interpret and successfully identify students' difficulties. The main purpose of the present study was to examine mathematics teachers' geometrical figure apprehension in relation to their ability to construct geometrical proofs and to predict students' difficulties during the teaching of the specific topics. We therefore focused on whether teachers could apply Duval's theory of geometrical figure apprehension while proving specific geometrical tasks and also, on whether they could predict and interpret potential student difficulties and errors as it concerns these tasks.

2 Theoretical background

2.1 The teaching of geometry and geometrical proof in secondary education

Engagement with geometrical content promotes basic cognitive skills and contributes to the understanding of the world in which we live (Kuzle, 2022). The domain of teaching and learning of geometry is always of considerable international interest with many questions remaining unanswered in respect to the teaching methods and the respective learning outcomes. An ICME Monograph in 2018 was developed about the international perspectives on the teaching and learning of geometry in secondary schools.

Many studies have examined the connections between geometrical thinking and spatial abilities (Jones & Tzekaki, 2016). According to Battista et al. (2018) a spatial ability is divided into spatial visualization and property-based analytic reasoning. The first component involves mentally creating images of objects while the second component involves decomposing objects into their parts by using geometrical properties in order to specify how the parts are related. In general, there are various studies suggesting that a student's spatial ability is related to his/her geometrical performance (Tso & Liang, 2002). Panaoura et al. (2007) showed that the performance of elementary and secondary school[1] students in spatial ability tests could act as a predictor of their performance in geometrical tasks. Furthermore, some researchers have tried to relate spatial ability with geometrical figure apprehension and creativity in geometry (Gagatsis & Geitona, 2021; Gagatsis et al., 2022).

Geometry is to a large extent related with the understanding and the use of geometrical figures. Figures are mental entities which exist only based on definitions and properties defining them (Michael-Chrysanthou & Gagatsis, 2014). However, when teaching geometry, teachers might fail to give the essential attention to the structure of a figure, either due to time pressure or to their insufficient skills to draw a figure on the board with accuracy. Additionally, it might be due to teachers' lack of skills in proposing a variety of examples with sufficient generic characteristics on which to build all the properties of the figure.

Understanding and constructing a mathematical proof is crucial in the teaching of geometry. Proving is a fundamental part of mathematical learning as it involves conjecturing, generalizing and justifying and it requires students to think about mathematical ideas in a flexible way (Lesseig, 2016).

> «Axiomatic proof is a mathematical argument consisting of a sequence of connected statements in support of a mathematical claim, each statement logically deduced from previous statements and justified by a combination of given statements, axioms and previously proven theorems».
> (Winer & Battista, 2022, p. 3)

Hunte (2018) argues that when students engage in reasoning and proving they have the opportunity to develop a deeper conceptual understanding of mathematical content and appreciate the purpose of reasoning and proof in mathematics. During reasoning activities, students make sense of patterns or conjectures which eventually lead them to develop counter-arguments or proofs that support their sense making. Geometrical proof is related with reasoning based on geometrical properties. Geometrical reasoning is related with spatial reasoning, which has been previously mentioned, and it is necessary to examine the degree to which students' spatial reasoning depends on the use of properties (Battista et al., 2018).

The difficulties in the teaching and learning of proof are well recognized internationally (Miyazaki et al., 2016). Even undergraduate mathematics students face difficulties in understanding and constructing mathematical proofs (Zazkis & Zazkis, 2013). Winer and Battista (2022) found that the majority of students utilize sound reasoning in their oral explanations but struggle to capture that reasoning in their written proofs. As a way of supporting geometry proof reasoning Cheng and Lin (2008) developed a step-by-step reasoning strategy and found that this teaching strategy improves students' proving process. According to Mwadzaangati and Kazima (2019) the reports at African educational systems highlight teachers' lack of knowledge for teaching geometrical proofs as the main cause of students' weak-

1. The education system in Cyprus is divided into 6 years of primary education (grades from 1 to 6), 3 years of lower secondary education (*Gymnasio*, grades from 7 to 9), 3 years of upper secondary education (*Lykeio*, grades from 10 to 12), and subsequent university education.

ness in geometrical proof development. Research on students' engagement with proof has tended to focus on students' difficulties with logical ideas (Stylianides, 2018). Traditionally, in the school curriculum proof has been taught more widely in the context of Euclidean Geometry and is presented as a formal confirmation of statements that are considered to be true. A major component of the validity of students' geometrical proof is the sequence of logical deductions they make. However, the construction of a proof based on the geometrical properties presupposes the understanding of those properties and the ability to correlate them at a structure of geometrical reasoning. Duval (2007) argues that the most important part of constructing a logical proof is understanding the status or function of each proposition with a single deduction.

In respect to the teaching of geometry, a huge amount of research focuses on teachers' content knowledge and their respective teaching knowledge. As Ball et al. (2001) pointed out, very few studies have focused on this issue and the existing ones have sometimes provided some paradoxical results, as in Begle's paper (1979, cited by Ball et al., 2001) in which he concluded that greater mathematics subject knowledge could be associated with a negative effect on students' achievement. Shulman (1986) introduced the notion of pedagogical content knowledge (PCK) to complement subject content knowledge, and based on this idea, various refinements have been made to describe knowledge that is really needed to teach mathematics. Hill et al. (2008) have introduced the notion of knowledge of contents and students (KCS) and of knowledge of content and teaching (KCT) to organize mathematical knowledge for teaching (MKT). Working from the didactic research perspective, the related notion of didactical content knowledge (Houdement & Kuzniak, 1996, 2001) was similarly introduced for the purpose of investigating the knowledge of the didactics of mathematics teachers need in the classroom. Teachers seem to look at students' performance through some preconceived notions of geometry and this leads them to adopt different attitudes in respect to the difficulties they encounter. In the case of the geometrical proof, Fuglestad and Goodchild (2009) examined teachers' knowledge about proof, concluding that some teachers do not appear certain about the nature and necessity of a proof. Cirillo (2011) indicated that even a beginning teacher with a strong mathematics background cannot be well prepared to teach proof.

2.2 The theory of geometrical figure apprehension of Raymond Duval

Duval (1993) developed a theory concerning a semiotic approach to learning mathematics. Key notions in his theory are the registers of semiotic representations, the transformations between the representations and the cognitive functioning of thought. He has also developed an important theory relating to geometry teaching and learning and more specifically, to the apprehension of the geometrical figure (Duval, 1995, 1998, 2005).

Geometrical figures are representations possessing a central role in geometrical activity. A figure merges three semiotic representations: magnitude, shape configurations and words naming the given properties. According to Duval (2005), the crucial issue in the learning of geometry is the separation between magnitude and visualization, because magnitude causes visual illusions and wrong perceptual estimation regarding the relations between figural units. Thus, the difficulties most students face are due to a cognitive gap between two opposite ways of looking at figures and recognizing what they stand for: the natural perceptive way as for any visual representation of material objects or spatial organization (images, diagrams, plans etc.) and the mathematical way for reasoning, defining, solving problems or proving. Perceptual appreciation is sometimes misleading for the recognition of geometrical properties and, therefore, for the recognition of the geometrical objects represented. On the other hand, visualization is independent from magnitude and concerns only shape discrimination and configuration (Duval, 1995). Visualization is the simultaneous and immediate apprehension of a configuration as a whole.

Duval distinguishes four kinds of geometrical figure apprehension: the perceptual, the sequential, the discursive, and the operative. For a drawing to function as a "geometrical figure", perceptual apprehension and at least one of the other three types of apprehension must be elicited.

– *Perceptual apprehension* is a person's ability to name figures and recognize several sub-figures in a plane or depth. One's perception of what the figure shows is determined by figural organization laws and pictorial cues. For instance, in **Figure 1** below it is possible to look at the figure ABCEDF in two different ways (Gagatsis, 2015).

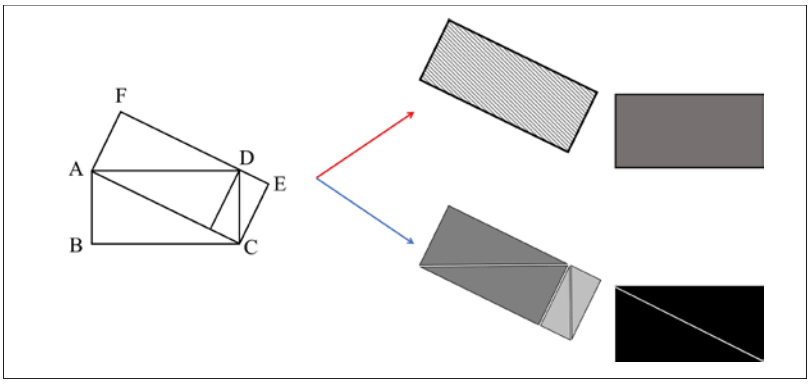


Figure 1. Example of the different ways of perceptual apprehension.

Particularly, the perceptive way of visual recognition focuses exclusively on the most global shape or closed outline, according to the principles stated by the Gestalt theory, thus the recognition of other possible reconfigurations is excluded. The perceptive way is activated and reinforced when figures are used as objects that can be empirically observed and it can either help or inhibit the heuristic recognition.

– *Sequential apprehension* refers to a person's ability to either construct a figure or describe the construction of a figure. The organization of the elementary figural units does not depend on perceptual apprehension but rather, on technical constraints and mathematical properties.
– *Operative apprehension* is a form of visual processing that gives an insight into a problem solution when looking at a figure and depends on the various ways of modifying a given figure. One way is the mereologic way that refers to the division of the whole given figure into parts and the combination of them in another figure or sub-figures (reconfiguration). The mereologic way is a common procedure used in geometrical tasks concerning, for example, the area of the figures. Within the operative apprehension the given figure becomes a starting point to explore other configurations that also stem from the applications of other visual operations: the optic and the place way. The optic way is related to making the figure larger or narrower, or slant and the place way refers to the position or orientation variation of a figure. Finally, one important way of operative apprehension of a geometrical figure is the use of auxiliary lines in order to prove a geometrical task. It is important to note that, generally speaking, the proof of a geometrical task is more difficult, when the auxiliary lines are outside of the initial figure (Levav-Waynberg & Leikin, 2009).
– *Discursive apprehension* deals with the valid use of properties for deducing. A figure is seen in relation to denomination or a hypothesis that make certain properties explicit. Perceptual apprehension cannot determine the mathematical properties represented in a drawing (Duval, 1995), so some mathematical properties must be given through speech (denomination and hypothesis). The absence of denomination and hypothesis in a drawing makes it an ambiguous representation and, thus, the properties that are seen are not the same for everyone (Duval, 1995). Discursive apprehension is when an individual is using argumentation or discourse in order to prove the mathematical properties of the respective figure. This type of apprehension is necessary as an individual cannot determine mathematical properties through perceptual apprehension.

At many of our previous studies, it has been demonstrated that the different types of geometrical figure apprehension are closely related (Gagatsis, 2015; Gagatsis et al., 2015). In fact, a structural model has been verified for secondary school students (13-17 years old), that includes all four types of geometrical figure apprehension (perceptual, sequential, operative and discursive). In other words, the term "structural model" is used based on the statistical analysis known as SEM (Structural Equation Modelling), according to which a theoretical model is confirmed through the use of confirmatory factor analysis. It is worthwhile noticing that this model has the same structure for two different experimental populations, that is for lower secondary school students (13-15 years old) and upper secondary school students (16-17 years old). A similar structural model has been verified for primary school students (10-12 years old), which includes three types of geometrical figure apprehension (perceptual, discursive and operative) (Gagatsis et al., 2015). This result is important because despite the differences in students' knowledge and skills in respect to the three different populations and the differences between the geometry tasks proposed to the three populations, the structural models resulting from the SEM statistical analysis are similar. Moreover, Torregrosa and Quesada (2009) focused on reasoning in which discursive and operative apprehensions are coordinated in order to generate a proof. We believe that their claim is related to our experimental "structural model" and to its relation to the proofs generated by students and/or teachers.

3 Methodology

3.1 Course content

The research subjects were 42 secondary school mathematics teachers (either lower or upper secondary education) participating in an intensive pedagogical course. All teachers had a Bachelor's degree in Mathematics from a variety of different Universities in Cyprus, Greece, other European countries and the US. Additionally, the majority also had a Master's degree in Pure and/or Applied Mathematics and in Mathematics Education (Didactics of Mathematics). Furthermore, it is important to note that some of them were also PhD holders in different domains of Mathematics. Almost all of the teachers had experience in teaching mathematics at secondary schools in Cyprus. Before the attendance of the specific training course they were working as mathematics teachers at private schools or as non-permanent staff at public schools and they attended the specific course in order to become permanent teaching staff.
The course was offered within the Mathematics Education Postgraduate Programme by the Cyprus Pedagogical Institute in cooperation with the Department of Education of the University of Cyprus. Specifically, the course content included a brief review of some concepts and methods of research in mathematics education as well as results of different published research papers with interpretations of students' errors in tasks of algebra, analysis and geometry. A part of the course covered the teaching of geometry by theoretically presenting Duval's theory on figure apprehension. Special emphasis was placed on the teaching and learning of geometry with detailed references to geometrical figure apprehension as well as other concepts (Duval's semiotic theory, Fischbein's theory of figural concepts, Brousseau's theory on didactic situations and Van Hiele's model of geometrical reasoning). A set of concepts, methods and theories relating to didactics of mathematics have been developed which delimit its field of validity (Brousseau, 1997). Theories regarding concepts and methods of didactics of mathematics are evident in various research studies concerning different didactic situations and various mathematical concepts. The concept of didactic contract, for instance, has been introduced by Brousseau and it refers to the various teaching situations where there is an implicit agreement between a teacher and a student on what is handled by each "partner" as well as on their responsibilities, commitments etc. The role of the didactic contract is to regulate the interactions

between the teacher and the student in terms of a piece of knowledge. The teacher respects the contract by teaching and assessing and the student respects the contract if she/he completes the assessment, does the homework and tries to understand the teacher's expectations in order to fulfil them. Brousseau (1989) also studied the contract that students must keep while solving problems. Those problems are usually ordinary problems which receive only one answer by using all the data through familiar processes and the solution of a geometrical problem must be found either by applying a theorem or property or it derives directly from the figure. Due to their compliance with contract terms, students work in a typical, formal and non-realistic way (Verschaffel et al., 2000). Figure 2 is an indicative simple example of a geometrical task where the concept of the didactic contract could interfere (Panaoura-Maki, 2007). Most students find the obvious solution $\overline{AC}$ = 6 cm as a square side. Nevertheless, many of them are not satisfied with the obvious solution and applied the Pythagorean theorem to find it because, according to the didactic contract, a mathematical solution is acceptable only when it results from the application of an algorithm, a theorem or a mathematical property. Real learning does not happen through a blind obedience to the terms of the contract, on the contrary it happens when the didactic contract is breached (Brousseau, 1989).

On the right triangle ABC, $\overline{BC}$ = 10 cm and $\overline{AB}$ = 8 cm. ACDE is a square ($\overline{CD}$ = 6 cm). Find the length of segment AC.

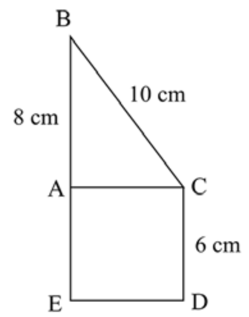


Figure 2. A geometrical task in which the didactic contract can interfere.

3.2 The written test

Participants completed a written test consisting of two parts (see Attachment 1): part A included six simple geometrical tasks for lower secondary school students (12-14 years old) and part B included three questions which aimed to examine participants' ability to predict students' mistakes and difficulties. The tasks included in the second part were three geometry propositions concerning the equality of two heights or of two medians and/or two bisectors of a triangle. In all three cases the triangle was isosceles. These three tasks had an almost identical wording but the solving difficulty differed considerably. In other words, one could argue that there is congruence among the way the three tasks were presented. We concentrated on identifying the relations between the solutions teachers had given to the tasks at both parts of the test and their interpretations concerning the students' possible difficulties to solve the tasks, indicating geometrical figure apprehension. The tasks were introduced to teachers with the following text: «Present the correct answer at each task and explain how the students will work and their possible mistakes. Present and explain what concepts or theories of didactics intervene in their solution».

3.3 Results' analysis

The analysis of the results was divided into three stages: a) a qualitative analysis of the way teachers analyzed a priori the concepts of didactics which are involved in the solutions of the tasks, b) a quantitative analysis of participants' solutions and predictions about students' difficulties and mistakes, c) an implicative statistical analysis in order to examine the interrelations among their solutions and their

respective teaching expectations. Implicative statistical analysis was founded by Gras' for his doctoral thesis on objectives of Didactics of Mathematics (Gras et al., 2013; Gras et al., 2008). The software program that implements the method is called CHIC and there are several versions of it (for this study version 6.1 was used). The results are produced in algebraic or diagrammatic form. In this research two diagrams are presented, which are the most important: the implicative graph and the similarity diagram. In the implicative graphs there are arrows between the different tasks (variables). If an arrow exists between variable A and variable B, A → B, it means that success to A implies success to B. On the other hand, in the similarity diagram, if two variables are connected with two vertical lines, it means that the two tasks are processed in a similar way.
Gras' method is particularly effective in the use of representations and in particular, in the phenomenon of compartmentalization of representations in mathematics teaching. Compartmentalization is identified as a cognitive difficulty when an individual attempt to interpret and move back and forth between different types of representations in mathematics (Duval, 2002). Vinner and Dreyfus (1989) extended the concept by arguing that compartmentalization arises when an individual has two divergent, potentially contradictory schemes in his/her cognitive structure. They further stated that inconsistent behavior is an indication of compartmentalization.
The phenomenon of compartmentalization has been revealed through implicative statistical analysis in different mathematical concepts. In these studies, students tend to distinguish the mathematical tasks according to their initial representation. Some studies focus on the use of the geometrical model of the number line regarding addition and subtraction of whole numbers (Gagatsis & Shiakalli, 2004), while others focus on the use of representations in functions (Gagatsis et al., 2006). A compartmentalized way of thinking was evident in both primary and secondary school students. In all cases the implicative statistical analysis played an essential role in revealing the phenomenon of compartmentalization which negatively affects students' learning of mathematics.
For the scoring of the written test, we used the following marking scheme for both teachers' solutions and interpretations: A 5-point scale of scores between 0 and 1 (0; 0,25; 0,5; 0,75; 1) were assigned according to the level of correctness of the solution given or the level of relevance of the interpretation given. Two researchers independently graded the tests in order to cross-check results and give reliability to the scoring.

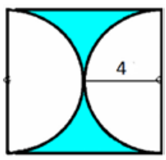
4 Results

4.1 Qualitative analysis of teachers' predictions about the tasks

Before the presentation of the analysis of each task separately, it is important to note that most teachers identified the following concepts or theories of didactics as factors intervening in the correct solution, i.e. all types of geometrical figure apprehension (perceptual, sequential, operative, discursive), spatial ability, spatial visualization and didactic contract. Below, we present and analyze each task along with teachers' working and comments.

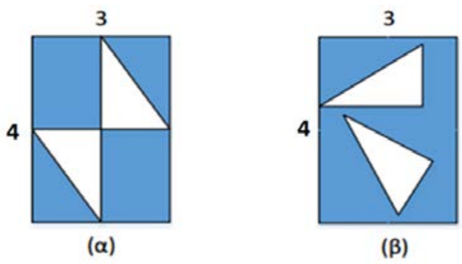
4.1.1 Part A

T1. In the figure below, the two semicircles within the square have a radius of 4 cm. Calculate the shaded area. Show the way you have worked.



In T1 perceptual and operative apprehension of the figure are necessary in order to solve the task, as well as knowledge of the area of a circle and properties of a square. The majority of teachers identified that perceptual apprehension is necessary in order to identify the subfigures in the given figure, i.e. to recognize the square and the two semi-circles. Then, teachers identified the need for operative apprehension of the geometrical figure; more specifically, teachers mentioned that students should have to perform mereologic modification or simple transformation of the figure in order to realize that the two semi-circles make one circle. Some teachers mentioned that students might not be able to recall the formula for the area of a square and a circle. A few teachers also mentioned that the concept of the didactic contract might act as a possible barrier, explaining that students may try to find the area of each semi-circle separately instead of directly finding the area of the circle.

T2. Two equal triangles are placed on a rectangle with sides 3 and 4, as shown in the figures below.
(i) Calculate the shaded area in figure (α).
(ii) In figure (β), the triangles are placed differently. What part of the rectangle do the triangles cover? Please explain the way you worked.

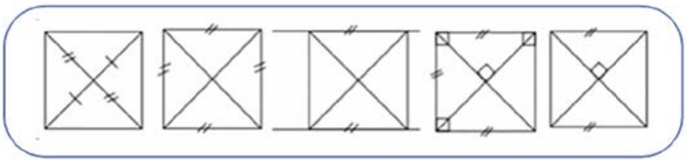


In both (i) and (ii) of T2, perceptual and operative apprehension of the figure are necessary. As far as operative apprehension is concerned, place way modification of the figure is necessary a student to realize that the two triangles cover ¼ of the area of the rectangle in both parts (i) and (ii) of the task. Spatial ability also intervenes in this task, especially for part (ii) where the two triangles change position.
Initially, all teachers mentioned that perceptual apprehension is needed for students to realize the dimensions of the two sides of the triangles. Additionally, all teachers recognized that operative apprehension of the geometrical figure, and more specifically mereologic modification of the figure, is necessary in order for students to realize that the two triangles cover the ¼ of the area of the rectangle. As far as part (ii) of the task is concerned, teachers identified that students might think that the shaded area changes and this again has to do with perceptual apprehension of the geometrical figure given. Most of the teachers also mentioned that spatial ability intervenes too, in the sense that students should understand that despite the fact that the two triangles change position from figure (α) to (β), the area does not change.

T3. How many triangles are there in the figure below?

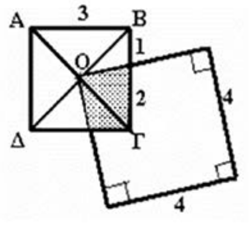
For this task, perceptual and operative apprehension of the figure are necessary. Mereologic modification of the figure would give a student the opportunity to split and reunite the subfigures and thus, correctly count the total number of triangles present. Most teachers replied that it is through perceptual apprehension that students can initially identify the more "obvious" triangles in the given figure. Then, most teachers agreed that operative apprehension of the geometrical figure intervenes, as students need to perform mereologic modification in order to identify the subfigures and also, in order to split and reunite them. That is, they will need to break the squares in triangles and also reunite subfigures to create new ones. Some teachers also mentioned that spatial abilities and spatial visualization is needed in order for students to also identify the biggest triangles, i.e. the ones that exist by uniting 4 small triangles.

T4. In the figure below, students are asked two questions:
a) What information is given for each one of the figures?
b) Please state, where possible, the exact nature of the quadrilateral.

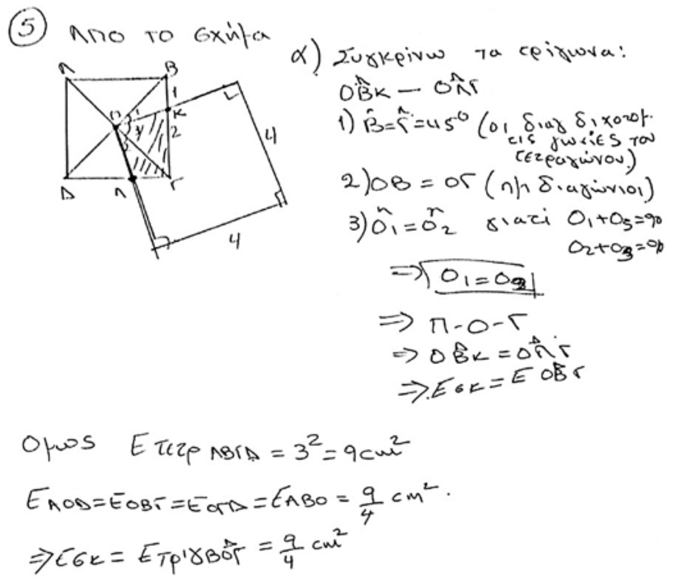


Although perceptual apprehension is needed for this task, it is through discursive apprehension that a student can correctly solve this task. This is because students need to prove by using their knowledge of the properties of a quadrilateral, the nature of each figure given, instead of using their perceptual apprehension only.
By solving T4, most of the teachers observed that the concepts of perceptual and discursive apprehension intervene in the solution of this task. In other words, teachers believe that the most possible difficulty can be that students take into consideration the representation of the figure and not the properties presented. Therefore, students will not prove the nature of each quadrilateral using mathematical properties and theorems but, instead, they will make conclusions by using their perceptual apprehension only. For example, a teacher mentioned that, for the first quadrilateral, students can understand from the image that the angles of the figures are right whereas this should not be deduced by the figure but, instead, it should be proved. Overall, most teachers successfully observed that discursive and perceptual apprehension of the figure are necessary in order for students to make connections between each figure and the respective rectangle properties.

T5. In the following figure, O is the point of intersection of the diagonals of the square ΑΒΓΔ. Find the area of the shaded region.



T5 was a task with low success rate in both solution and interpretation. It is the most difficult task among the five of the first group of tasks and the third most difficult task of the eight tasks of the questionnaire. T5 requires perceptual, operative (place way modification) and discursive apprehension of the geometrical figure. First of all, teachers mentioned that perceptual apprehension might intervene so that students cannot be able to identify which triangles they need to compare. Additionally, teachers mentioned that operative apprehension, and more specifically mereologic modification, is necessary for the solution of this task. Teachers identified as a possible difficulty that students will not be able to recognize the equality among the two small triangles and will therefore not be able to move forward with a solution. This also has to do with place way modification; some teachers did mention this but instead of using the term "place way modification" they described it as "change of position". However, the crucial observation that immediately gives the solution to the task is the equality of the two triangles ΟΒΚ and ΟΛΓ, as shown by the solution of the teacher in Figure 3. This equality, however, of the two triangles is evident from a mental rotation of the triangle ΟΒΚ in such a way that it coincides with the triangle ΟΛΓ.



Translation:

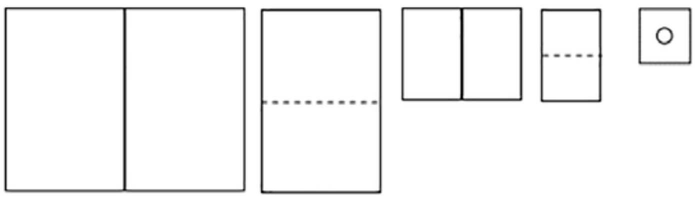
From the figure, a) I compare triangles: ΟΒΚ – ΟΛΓ
1) angle B = angle Γ = 45° (diagonals bisect the angles of the square)
2) OB = ΟΓ (diagonals)
3) angle O_1 = angle O_2 because $O_1 + O_3 = 90°$ and $O_2 + O_3 = 90°$
→ angle O_1 = angle O_2
→ side – adjacent angles equal
→ triangle ΟΒΚ = triangle ΟΛΓ
→ Area of shaded region = Area of ΟΒΓ.

However, Area of square ΑΒΓΔ = 3² = 9 cm².
Area of ΑΟΔ = Area of ΟΒΓ = Area of ΟΓΔ = Area of ABO = $\frac{9}{4}$ cm².
→ Area of shaded region = Area of triangle ΒΟΓ = $\frac{9}{4}$ cm².

Figure 3. An example of a teacher's correct solution.

Thus, spatial ability is also necessary, and more specifically spatial visualization and flexibility of closure factor, in order for students to imagine rotations of objects (subfigures) and thus, identify the subfigures necessary for the solution, respectively. On the other hand, the concept of didactic contract might also intervene in two different ways but few teachers correctly identified and explained this. Firstly, many teachers dealt with the geometrical proof of the equality of the two triangles based on the data given in the figure. But a significant number of them did not notice that this equality is evident from a turn of the shape so that one triangle could identify with the other. Secondly few teachers mentioned that the length of the side of the bigger square (4 cm) is not necessary for the solution of the problem but none of the teachers realized that those data are not compatible with the rest of the data given in the problem. So, some of them try to solve the task ignoring the length 4 cm. A few teachers also referred to spatial ability as a concept intervening in the solution by explaining that students will need to mentally modify the figure given in order to identify that the shaded area is ¼ of the total area. A few teachers also mentioned the concept of didactic contract intervening in the solution. Teachers explained it in two ways: a) students will try to use all data given whereas not all data is needed in order to reach a solution, b) students will try to calculate the shaded area by deducting the area of the small square from the large square just for the sake of performing an operation.

T6. Maria folds a piece of paper in half and then repeats the same process 3 times. Then, she opens a hole on the piece of paper. If she unfolds the piece of paper, how many holes will the paper have? Please show your working and explain how you reached your answer.



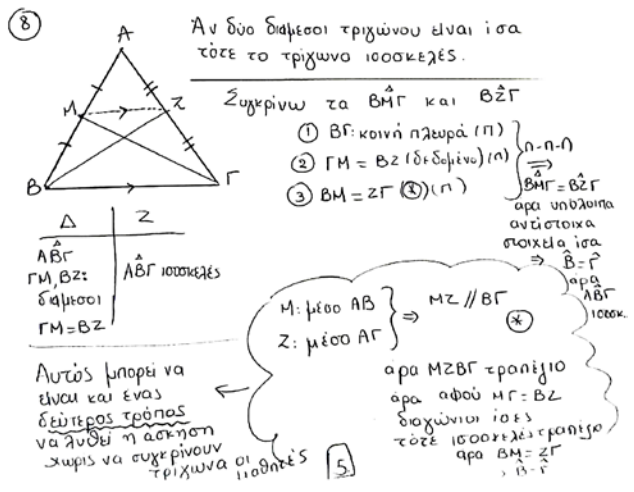
In T6, perceptual and sequential apprehension of the figure is needed together with spatial ability, and more specifically spatial visualization. Most of the teachers mentioned that students can face difficulty in two operations: recognizing symmetry and mentally modifying the figure. Some associated this difficulty to the concepts of spatial ability and spatial visualization while others described this as a difficulty in sequential and discursive apprehension. Few teachers predicted that students can solve the task through an algebraic approach, i.e. by using the theory of a geometrical sequence.

4.1.2 Part B

Part B of the written test consisted of the following three questions:

TR1. Could we argue that a triangle is isosceles, if two of its heights are equal?
TR2. Could we argue that a triangle is isosceles, if two of its medians are equal?
TR3. Could we argue that a triangle is isosceles, if two of its bisectors are equal?
Compare the three questions on the basis of how difficult it is for students to prove each one of them and explain the reasons why, if any, there is a difference in difficulty. Which didactic concepts intervene?

All three tasks of Part B involved proving processes. For TR2 and TR3, which were the tasks with the lowest success percentages in both solution and interpretation, we first present an example of a teacher's correct solution and then we analyze the tasks. Figure 4 and Figures 5a-5b present an example of a teacher's correct solution of TR2 and TR3 respectively.



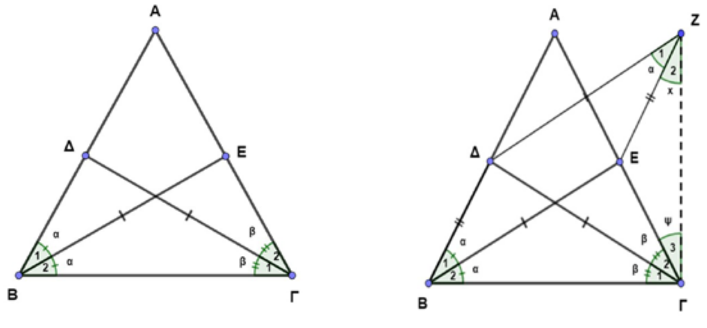
Translation of TR2 solution:

I compare triangles ΒΜΓ and ΒΖΓ
1) ΒΓ: common side
2) ΓΜ = ΒΖ (as given)
3) ΒΜ = ΖΓ*
*: because M is midpoint of ΑΒ and Ζ midpoint of ΑΓ, therefore ΜΖ is parallel to ΒΓ and therefore, ΜΖΒΓ is a trapezium. So, since ΜΓ = ΒΖ, the diagonals are equal then it is an isosceles trapezium and therefore ΒΜ = ΖΓ and angle Β = angle Γ.

Since 1), 2), 3) are true then because of three sides equal, triangle ΒΜΓ = triangle ΒΖΓ and so angle Β = angle Γ, therefore triangle ΑΒΓ is isosceles.

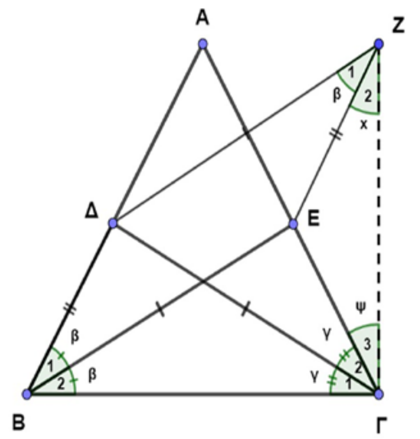
Figure 4. An example of a teacher's correct solution of the task TR2.

In Figures 5a-5b below, because TR3 required a complex and lengthy solution, we only show the figure constructed by the teacher and we directly present the translation of the solution.



Initial figure constructed. Final modified figure.

Figure 5a. The transformation of the initial figure of the triangle in task TR3.



In triangle $AB\Gamma$, I bring the bisectors BE and $\Gamma\Delta$ of angles B and Γ. The bisectors are equal $BE = \Gamma\Delta$.
BE bisector, so $\hat{B}_1 = \hat{B}_2 = \beta$.
$\Gamma\Delta$ bisector, so $\hat{\Gamma}_1 = \hat{\Gamma}_2 = \gamma$.

I construct point Z such that ΔBEZ is a parallelogram. From the properties of the parallelogram:
- Opposite sides equal $B\Delta = EZ$ and $\Delta Z = BE = \Gamma\Delta$.
- Opposite angles equal, $\Delta\hat{B}E = \Delta\hat{Z}E = \alpha$.

We hypothesize that triangle $AB\Gamma$ is not isosceles and that $\hat{B} < \hat{\Gamma}$. Then, $\hat{B} = 2\beta < 2\gamma = \hat{\Gamma}$ and so $\beta < \gamma$.

Assume that $\hat{Z}_2 = x$ and $\hat{\Gamma}_3 = \psi$.
Triangle $\Delta Z\Gamma$ is isosceles because $\Delta Z = \Delta\Gamma$ and so the angles of its base are equal $\rightarrow \Delta\hat{\Gamma}Z = \Delta\hat{Z}\Gamma \rightarrow \gamma + \psi = \beta + x$.

We hypothesized that $\beta < \gamma$ so it should be true that $x > \psi$ for $\gamma + \psi = \beta + x$ to also be true. This means that $E\Gamma > EZ$ and since $EZ = B\Delta$ then $E\Gamma > B\Delta$.

If we observe triangles $\Delta B\Gamma$ and $EB\Gamma$ we have:
- $B\Gamma$ common side
- $BE = \Gamma\Delta$ (given)
- $E\Gamma > B\Delta$ (χ).

From the above and since side $E\Gamma$ is bigger than $B\Delta$, then it is true that $\beta > \gamma \rightarrow 2\beta > 2\gamma \rightarrow \hat{B} > \hat{\Gamma}$, which leads us to a contradiction since we initially hypothesized that $\hat{B} < \hat{\Gamma}$. I was led to a contradiction because of the wrong assumption I made that a triangle with two equal bisectors cannot also have two equal angles. Therefore, the triangle has two angles equal and therefore is isosceles.

Figure 5b. A solution of task TR3 (directly translated by the authors).

Task TR1, requires perceptual, sequential and discursive geometrical figure apprehension while Tasks TR2 and TR3 require all types of geometrical figure apprehension (perceptual, sequential, operative and discursive) as well as excellent knowledge of geometrical properties and theorems. Sequential apprehension is necessary in order for students to construct a correct figure. Perceptual apprehension would help them identify which figures they can use to find a solution path, whereas operative apprehension would enable students to modify the constructed figure and especially construct auxiliary lines. Of course, without discursive apprehension and excellent knowledge of geometrical properties and theory students would not be successful in making the necessary connections and building the argumentation needed for a successful proving procedure.

TR1, TR2, and TR3 differ in difficulty despite the fact that their wording does not directly reveal it. Almost all teachers agreed that the proof of TR3 was the most difficult one and the proof of TR1 the least difficult. TR1 can be proved directly by combining the data given with the relevant geometrical theory (triangle comparison). TR2 can be solved directly through triangle comparison however, it is necessary to use one additional theorem which refers to the congruence of the three medians of a triangle. TR2 can also be solved by using appropriate auxiliary lines. However, for the solution of TR3 students need to construct the figure correctly, then modify the figure by drawing appropriate auxiliary lines and use the proof by contradiction method. It is true that students are more acquainted with direct proving processes compared to proof by contradiction.

According to participants' answers, the concept of figure representation intervenes in all three tasks. Teachers agreed that students will find it difficult to correctly construct the figure, both mentally and on paper (sequential apprehension). Furthermore, their answers showed that they believed that perceptual apprehension would be needed for students to identify which subfigures to compare. Operative apprehension was also mentioned as a potential student difficulty because of the necessary mereologic modification for the construction of the figure (especially for TR3). This is clearly shown in Figure 5a above, which presents the necessary transformation of the figure, from its construction to the solution of the task.

From Figures 5a-5b, it is evident that students would have to translate correctly the wording of task TR3 into a figure but they have to realize that they could not solve the task unless they modified the figure. This modification is rather complex as it includes drawing auxiliary lines outside the figure constructed. According to Leikin and Elgrabli (2015) two factors determine the difficulty in auxiliary constructions: the location of the auxiliary lines (inside or outside the figure) and the number of lines needed for a solution to be identified.

4.2 Quantitative analysis of solutions and predictions

In order to analyze and compare teachers' solutions and interpretations, we used two metrics and their respective percentages. The first metric is the cumulative sum of the scores the teachers received while trying to solve a task. So, to give an example, in T1 all teachers (42 in total) gave a fully correct answer and therefore received 1 whole point from the 5-point scale (0; 0,25; 0,5; 0,75; 1). This gave a "cumulative score" of 42, which was then converted to a success percentage (in this example it is 100%). Another example is that of T2 where teachers' scores were added together and received a cumulative sum of 41,75 which gives a success percentage of 99,4% (calculated as $\frac{41,75}{42} \times 100$). The other metric is the cumulative sum of the scores teachers received from interpreting the tasks; this metric was again converted to a success percentage. In the next paragraphs, we therefore use the terms "cumulative sum – solution", "cumulative sum – interpretation", "success rate – solution" and "success rate – interpretation" to help the reader follow the analysis.

The quantitative analysis of teachers' solutions showed that, on average, the success rate, as defined above, in solving the tasks was 86,6%. If we look separately at the two parts of the written test, i.e. part A consisting of six tasks (coded as T1, T2, T3, T4, T5 and T6) and part B consisting of three questions (coded as TR1, TR2 and TR3), then we observe that teachers reacted better in the first six tasks (part A) compared to the question tasks (part B), with 95% and 69,6% success respectively. It is worth noting here that the questions in part B involved proving processes. Figure 6 presents the "success rate – solution" per task. It is important to note here that in most tasks, teachers received a score of either 0 or 1 from the 5-point scale.

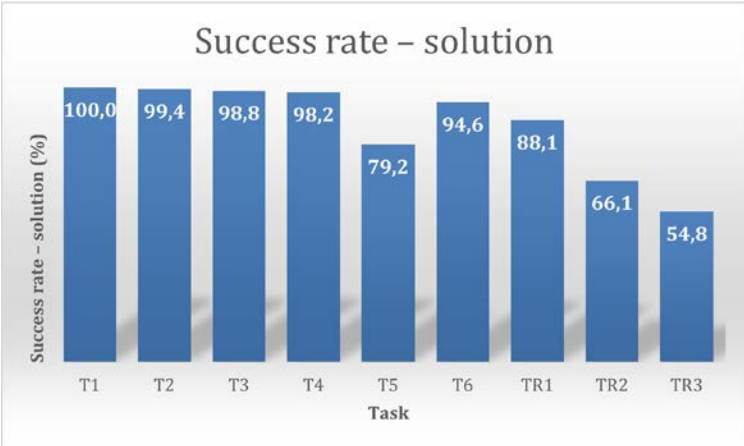


Figure 6. Success rate – solution, i.e. the cumulative sum of the scores teachers received in solving each task, converted to percentage.

Figure 7 presents the "success rate – interpretation" which was derived from the cumulative sum of the scores teachers received in the interpretation of each task. In other words, this metric shows that teachers predicted correctly the difficulties that students would face as well as the concepts and theories of didactics involved in solving each task.

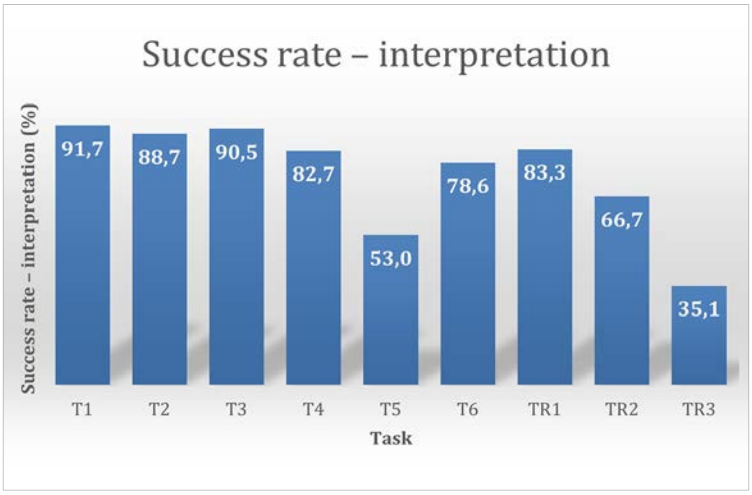


Figure 7. Success rate – interpretation, i.e. the cumulative sum of the scores teachers received in interpreting each task, converted to percentage.

Figure 7 shows that in all tasks, apart from TR3, the majority of teachers predicted quite successfully the difficulties students would face in solving or proving them as well as the theories and concepts of didactics intervening. In T5 approximately half of the teachers gave a correct interpretation. The tasks with the lowest success in interpretation were TR3, T5 and TR2. If we look at the results of the two parts of the written test separately, then we deduce that teachers were more successful in interpreting the tasks of the first part compared to the second.

By comparing the success rates in both solving and interpreting for each task, we observe similar results. Figure 8 presents this comparison.

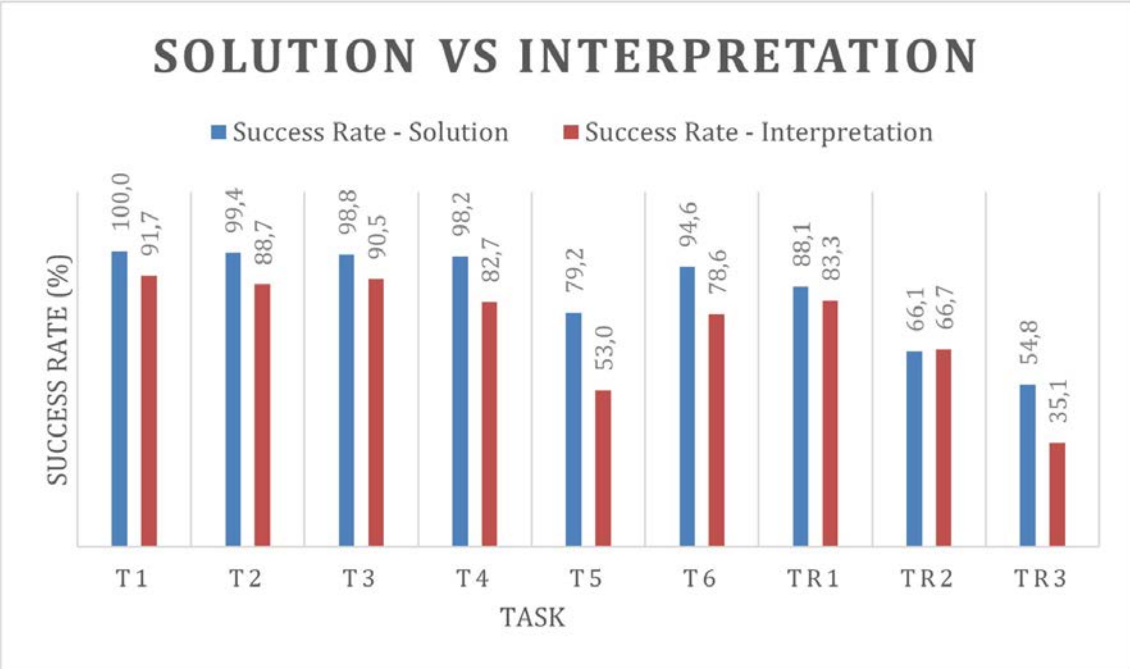


Figure 8. Success rate regarding solutions per task compared to success rate regarding interpretation per task.

Figure 8 summarizes our quantitative analysis by showing that except for TR2 where the success rates were almost equal (66,1% and 66,7%), teachers were more successful in solving the tasks than in interpreting them and actually being able to predict students' difficulties. TR3 was found to be the most difficult task either to solve or to interpret.

The largest discrepancies between solving and interpreting the tasks were found at tasks T5, TR3, T6 and T4. It is surprising to see task T6 and T4 in this category because, as is evident from Figure 6, it was not a difficult task for teachers to solve. The success rates regarding solutions of tasks T6 and T4 were 94,6% and 98,2% respectively. However, the success rates regarding interpretation of the same tasks were 78,6% and 82,7%.

4.3 Implicative statistical analysis

To summarize the similarity relations observed among the tasks, in terms of both the solutions and the interpretations teachers gave, we observe the phenomenon of compartmentalization between the tasks proposed to students. In particular, the similarity diagram (Figure 9) reveals five different groups of tasks, which are not related to each other. This is also evident in Table 1 below, which presents in detail the similarity between the variables. The first 10 similarity indexes seem to be relatively high. No general tendency was observed, as five distinguished groups of variables are created. The first group (Group 1) concerns the success in solving the first two tasks T1 and T2. Group 2 relates teachers' interpretations in Tint1, Tint2, Tint6 with their success in solving T6. Group 3 relates T3, T4, T5 and Tint5. The fourth group relates Tint3, Tint4, TR1 and TR1int. Finally, Group 5 is the group that is worth the most attention as far as the similarity of the variables is concerned; it relates teachers' solutions of tasks TR2 and TR3 with the interpretation they give for the same tasks.

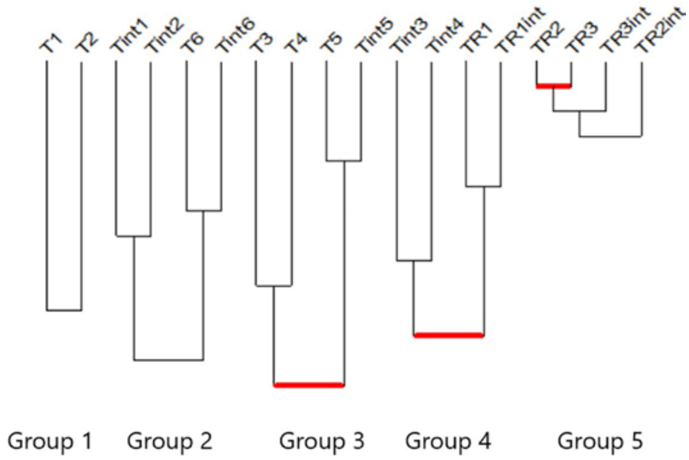


Figure 9. Similarity diagram.

Classification Level	Tasks or Variables	Similarity
1	(TR2 TR3)	0,969504
2	((TR2 TR3) TR3int)	0,90235
3	(((TR2 TR3) TR3int) TR2int)	0,821246
4	(T5 Tint5)	0,790061
5	(TR1 TR1int)	0,745479
6	(T6 Tint6)	0,59841
7	(Tint1 Tint2)	0,549653
8	(Tint3 Tint4)	0,539743
9	(T3 T4)	0,515061
10	(T1 T2)	0,5
11	((Tint3 Tint4) (TR1 TR1int))	0,128707
12	((Tint1 Tint2) (T6 Tint6))	0,107071
13	((T3 T4) (T5 Tint5))	0,0749481
14	((T1 T2) (((TR2 TR3) TR3int) TR2int))	0,00451588
15	(((T1 T2) (((TR2 TR3) TR3int) TR2int)) ((Tint3 Tint4) (TR1 TR1int)))	0,000212095
16	(((Tint1 Tint2) (T6 Tint6)) ((T3 T4) (T5 Tint5)))	3,81905e-05

Table 1. Levels of similarity (the range of values of the index of similarity is between 0 and 1).

In fact, tasks TR2 and TR3 present the greatest difficulties of all other tasks because their solution requires a transformation of the original shape using auxiliary lines TR2 and TR3. In addition, depending on the solution followed by the teachers (and/or students), the knowledge of triangles' equality criteria or even more specific knowledge is required, such as the property of the intersection of the triangle median (TR2) or the method of proof by contradiction (TR3). The second most important group of tasks is Group 4 ((Tint3 Tint4) (TR1 TR1int)), which relates the solution teachers give to TR1 with the interpretations they give to tasks TR1, T3 and T4. Finally, according to the teachers' answers, the solution of the tasks is not directly related with the variables of teachers' interpretation; we observe the couples of variables (T6, Tint6), (T5, Tint5), (TR1, TR1int) and Group 5 of variables (((TR2 TR3) TR3int) TR2int).
A similar phenomenon of compartmentalization is observed in the implicative graph, which is presented in Figure 10.

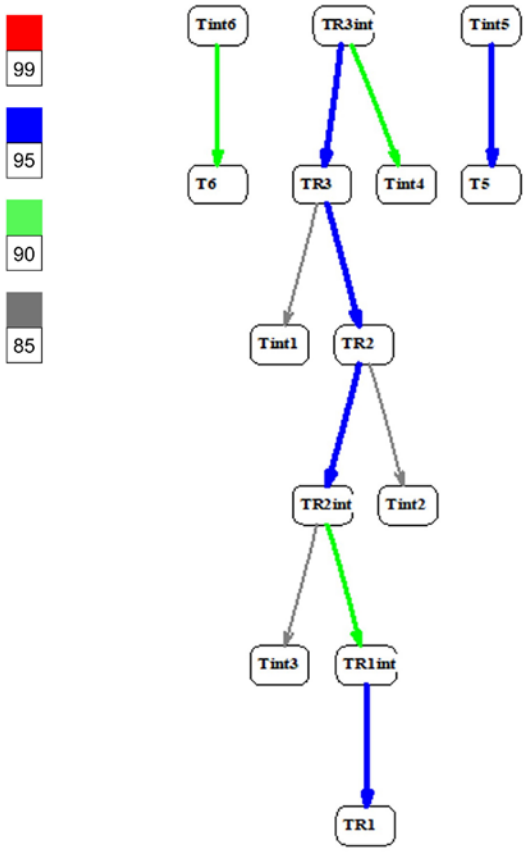


Figure 10. Implicative graph (the numbers on the side indicate the significance of the arrows).

There are three chains of implications; the first two are similar to the previously presented similarity groups Tint6 → T6, Tint5 → T5. Their meaning is that teachers who are giving a good interpretation of potential students' difficulties in these tasks, will give a correct solution to the tasks. For teachers who give a reasonable explanation of the potential students' difficulties related to a task, it is possible that they will give a correct solution to the specific tasks. This tendency mainly concerns the tasks related to the triangles, that is the tasks of the second part rather than the six tasks of the first part. The third chain of the implicative graph presents the most important group of the implicative relations, which is the group starting with the interpretation of task T3 at the top end of the diagram, i.e. TR3int. This chain is very important because it includes 10 variables: 6 variables are related to the tasks of the isosceles triangle, TR1, TR2, TR3, TR1int, TR2int, TR3int; it is also important that it includes four other variables related to teachers' interpretations, Tint1, Tint2, Tint3 and Tint4. To simply explain Figure 10, if a teacher interprets task T3 correctly, then this implies that the same teacher would give a correct solution and interpretation of various other tasks. An additional finding which occurs from the graph above is that at the top end of all three implicative chains, we find variables related to interpretations and not solutions of tasks.

5 Discussion

Through the present study we investigated whether a specific group of mathematics teachers, participating in an intensive pedagogical course, could implement various concepts or theories of didactics in interpreting school geometric tasks, especially involving geometrical proofs. We aimed to examine their success in solving the tasks in relation to their ability to interpret related student difficulties. We believed that their knowledge and skills on geometrical figure apprehension had to be examined in relation to their ability to predict and interpret students' mistakes and misunderstandings. A particularity of our research is that we have proposed the tasks TR1, TR2, TR3 that have an equivalent wording. Although it is not immediately obvious by the wording of the tasks, TR1, TR2, and TR3 differ in difficulty. Almost all teachers agreed that TR3 was the most difficult task to prove, while TR1 was the least difficult to prove. For all three tasks, teachers agreed that the concept of geometrical figure apprehension intervenes. To be more precise, teachers predicted that students would find it difficult to mentally construct the figure then be unable to construct it correctly on paper (sequential apprehension). Then, all teachers agreed that perceptual apprehension would be needed for students to identify which subfigures to compare. Operative apprehension was also raised as a possible difficulty students might face because they would have to mereologically modify the figure constructed. More specifically, operative apprehension, i.e. modification of the figure, is most difficult for TR3. In other words, it seems that the teachers realized the importance of the theory of the geometrical figure apprehension.

As it is obvious from the quantitative analysis of teachers' solutions and predictions presented in par. 4.2, although teachers might solve a task, they might not be able to correctly predict the difficulty a student might face in solving the task or the theories and concepts of didactics intervening in its solving process.

Additionally, from the implicative statistical analysis of teachers' solutions and predictions presented in par. 4.3, we can deduce that if a teacher interprets the most difficult task T3 correctly, i.e. correctly predicts students' difficulties as well as the theories and concepts of didactics intervening in solving this task, then this implies that the same teacher would give a correct solution and interpretation of various other tasks. In other words, the teachers' capability in giving proofs in the three classical tasks TR1, TR2, TR3 in Euclidian geometry is strongly related to their interpretations not only to the above three tasks but also to four other tasks' interpretations. Furthermore, as already mentioned in the results section (see par. 4), at the top end of all three implicative chains we find variables related to interpretations and not solutions of tasks. This could mean that a teacher who could correctly interpret a task could also give correct solutions to tasks. In fact, the success to the six variables related to TR1, TR2 and TR3 is strongly related to the success of four other variables related to teachers' interpretations.

Results indicated that teachers realize the importance of geometrical figure apprehension theory and apply it in order to analyze the given geometrical tasks both in respect to the possible difficulties that students might face as well as in respect to the concepts and theories of didactics that intervene in the solution of each task. However, the qualitative analysis of teachers' responses showed that not all teachers give the same answers. In some tasks, some of the teachers see difficulties that students might face which are not in accordance neither with related studies nor with the view of the majority of the teachers participating in this study. In both analyses, one can observe the compartmentalization phenomenon in participants' responses, because various similarity groups and various implicative chains are formed. This can be partly explained by the difference in difficulty between the first group of tasks (task T1 to task T6) and the second group of tasks (TR1, TR2 and TR3). Indeed, in the solution process of the first six tasks, perceptual apprehension was required in all tasks, operative apprehension was needed in almost all tasks and basic discursive apprehension in some of the tasks. However, in all tasks of the second group, strong discursive apprehension was required. The three tasks of Part B of the written exam include three geometrical statements relating to two equal heights of a triangle (TR1), or two equal medians (TR2) or two equal bisectors (TR3), which require geometrical proof. Additionally, although the three tasks seem almost identical in their wording (semantic congruence), their difficulty increases as we move from the first to the second and the third task. Success in tasks TR1, TR2 and TR3 which have to do with the isosceles triangle, implies success in four variables of interpretation (i.e. correctly predicting student difficulties as well as the concepts and theories of didactics intervening in the solution).

Our focus on the way teachers understand and implement theories and concepts of didactics of mathematics in analyzing student tasks highlighted two relevant points: on the one hand, such a focus can shed light to various facets of teaching and learning of geometry, on the other hand it shows the important role of courses about the teaching of mathematics and the development of cognitive and mathematical thinking. It could be deduced that teachers who correctly predict potential student difficulties can organize their teaching in a way that is more efficient by choosing, for example, more appropriate tasks. Additionally, an important finding of our study showed that a teacher giving a correct solution to a geometrical problem does not necessarily have a deep understanding and knowledge of the underlying theories and concepts of didactics. This finding could be taken into consideration in the preparation of prospective teachers of mathematics. We consider our study as a first important step towards investigating teachers' understanding of theories and concepts of didactics of mathematics and how it could impact students' teaching and learning of geometry and mainly the teaching and learning of geometrical proof. We need to gain insights into how teaching approaches might be improved so that students can develop a more secure understanding of proofs and proving. A further study including more geometrical tasks, solved and interpreted by a bigger sample of prospective mathematics teachers would prove to be another valuable addition not only to the existing literature in mathematics education but also to geometry teaching and learning.

References

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433–456). Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special?. *Journal of Teacher Education*, *59*(5), 389–407.

Battista, M. T., Frazee, L. M., & Winer, M. L. (2018). Analyzing the relation between spatial and geometric reasoning for elementary and middle school students. In K. Mix & M. Battista (Eds.), *Visualizing Mathematics. Research in Mathematics Education* (pp. 195–228). Springer.

Bergstrom, C., & Zhang, D. (2016). Geometry interventions for K-12 students with and without disabilities: A research synthesis. *International Journal of Educational Research*, *80*, 134–154.

Brousseau, G. (1989). Obstacles epistemologiques, conflits socio-cognitifs et ingénierie didactique. In N. Bednarz & C. Carnier (Eds.), *Construction des savoirs* (pp. 277–285). Agence d'Arc.

Brousseau, G. (1997). *Theory of didactical situation in Mathematics: Didactique des Mathématiques 1970-1990.* Kluwer Academic Publisher.

Cheng, Y. H., & Lin, F. L. (2008). A study on the left behind students for enhancing their competence of geometry argumentation. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education and PME-NA XXX* (Vol. 2, pp. 305–312). PME.

Chinnappan, M., White, B., & Trenholm, S. (2018). Symbiosis between subject matter and pedagogical knowledge in Geometry. In P. Herbst, U. H. Cheah, P. R. Richard & K. Jones (Eds.), *International Perspectives on the Teaching and Learning of Geometry in Secondary Schools. ICME -13 Monographs* (pp. 145–161). Springer.

Cirillo, M. (2011). I'm the Sherpa guide: On the learning to teach proof in school mathematics. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 241–248). PME.

Cirillo, M. (2018). Engaging students with non-routine geometry proof tasks. In P. Herbst, U. H. Cheah, P. R. Richard & K. Jones (Eds.), *International Perspectives on the Teaching and Learning of Geometry in Secondary Schools. ICME -13 Monographs* (pp. 283–300). Springer.

Duval, R. (1993). Registros de Representación Semiótica y Funcionamiento Cognitivo del Pensamiento. *Annales de Didactique et de Sciencies Cognitives*, *5*, 37–65. (Translated into Spanish for educational purposes by F. Hitt and A. M. Ojeda, Departamento de Matemáticas Educativa CINVESTAV-IPN, México).

Duval, R. (1995). Geometrical Pictures: Kinds of representation and specific processes. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematical education* (pp. 142–157). Springer.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st century* (pp. 37–51). Kluwer Academic.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education*, *1*(2), 1–16.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie : Développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de Didactique et de Sciences Cognitives*, *10*, 5–53.

Duval, R. (2007). Cognitive functioning and understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in School* (pp. 135–161). Brill.

Fuglestad, A. B., & Goodchild, S. (2009). I thought it was a proof. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 379). PME.

Fujita, T., & Jones, K. (2014). Reasoning and proving in geometry in school mathematics textbooks in Japan. *International Journal of Educational Research*, *64*, 81–91.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometrical construction and proving activities: A case of cognitive unity?. In M. F. Pinton & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 9–16). PME.

Gagatsis, A. (2015). Explorando el rol de las figuras geométricas en el pensamiento geométrico. In B. D'Amore & M. I. Fandiño Pinilla (Eds.), *Didáctica de la Matemática – Una mirada internacional, empírica y teórica* (pp. 231–248). Universidad de la Sabana.

Gagatsis, A., Elia, I., Geitona, Z. Deliyianni, E., & Gridos, P. (2022). How could the Presentation of a Geometrical Task Influence Student Creativity?. *Journal of Research in Science, Mathematics and Technology Education*, *5*(1), 93–116. https://doi.org/10.31756/jrsmte.514

Gagatsis, A., Elia, I., & Mousoulides, N. (2006). Are registers of representations and problem-solving processes on functions compartmentalized in students' thinking?. *Revista Latinoamericana de Investigación en Matemática Educativa, Special Issue 2006*, 197–224.

Gagatsis, A., & Geitona, Z. (2021). A multidimensional approach to students' creativity in geometry: spatial ability, geometrical figure apprehension and multiple solutions in geometrical problems. *Mediterranean Journal for Research in Mathematics Education*, *18*, 5–16.

Gagatsis, A., Michael-Chrysanthou, P., Deliyianni, E., Panaoura, A., & Papagiannis, C. (2015). An insight to students' geometrical figure apprehension through the context of the fundamental educational thought. *Communication & Cognition*, *48*(3–4), 89–128.

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, *24*(5), 645–657.

Gras, R., Régnier, J. C., Marinica, C., & Guillet, F. (Eds.) (2013). *L'analyse statistique implicative. Méthode exploratoire et confirmatoire à la recherche de causalités*. Cépaduès Editions.

Gras, R., Suzuki, E., Guillet, F., & Spagnolo, F. (Eds.) (2008). *Statistical Implicative Analysis. Theory and Applications.* Springer Berlin.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Houdement, C., & Kuzniak, A. (1996). Autour des stratégies utilisées pour former les maîtres du premier degré en mathématiques [About strategies used to train primary school teachers in mathematics]. *Recherches en Didactique des Mathématiques*, *16*(3), 289–322.

Houdement, C., & Kuzniak, A. (2001). Pretty (good) didactical provocation as a tool for teachers' training in Geometry. In J. Novotná (Ed.), *Proceedings of CERME2* (pp. 292–304). Praha Charles University.

Hunte, A. (2018). Opportunities for reasoning and proving in Geometry in secondary school textbooks from Trinidad and Tobago. In P. Herbst, U. H. Cheah, P. R. Richard & K. Jones (Eds.), *International Perspectives on the Teaching and Learning of Geometry in Secondary Schools. ICME -13 Monographs* (pp. 39–58). Springer.

Jones, K. (2000). Teacher knowledge and professional development in Geometry. *Proceedings of the British Society for Research into Learning Mathematics*, *20*(3), 109–114.

Jones, K., & Tzekaki, M. (2016). Research on the teaching and learning of geometry. In A. Gutierrez, G. Leder & P. Boero (Eds.), *The second handbook on the psychology of mathematics education: The journey continues* (pp. 109–149). Sense.

Kuzle, A. (2022). The teaching of geometry in primary education: Is geometry still neglected in school mathematics?. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti (Eds.), *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education* (pp. 734–741). Free University of Bozen-Bolzano and ERME.

Leikin, R., & Elgrabli, H. (2015). Creativity and expertise: The chicken or the egg? Discovering properties of geometry figures in DGE. In K. Krainer & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1024–1031). ERME.

Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics*, *66*, 349–371.

Lesseig, K. (2016). Conjecturing, generalizing and justifying: building theory around teacher knowledge and proving. *International Journal for Mathematics Teaching and Learning*, *17*(3), 1–31.

Levav-Waynberg, A., & Leikin, R. (2009). Multiple solutions to a problem: A tool for assessment of mathematical thinking in geometry. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the sixth conference of the European Society for Research in Mathematics Education (CERME6)* (pp. 776–785). Institut national de recherche pédagogique.

Manizade, A., & Martinovic, D. (2018). Creating profiles of geometry teachers' PCK. In P. Herbst, U. H. Cheah, P. R. Richard & K. Jones (Eds.), *International Perspectives on the Teaching and Learning of Geometry in Secondary Schools. ICME -13 Monographs* (pp. 127–144). Springer.

Michael-Chrysanthou, P., & Gagatsis, A. (2014). Ambiguity in the way of looking at a geometrical figure. *Revista Latinoamericana de Investigación en Matemática Educativa*, *17*(4/1), 165–180.

Miyazaki, M., Fujita, T., & Jones, K. (2016). Students' understanding of the structure of deductive proof. *Educational Studies in Mathematics*, *94*, 223–239.

Mwadzaangati, L., & Kazima, M. (2019). An exploration of teaching for understanding the problem for geometric proof development: the case of two secondary school mathematics teachers. *African Journal of Research in Mathematics, Science and Technology Education*, *23*(3), 298–308.

Panaoura, G., Gagatsis, A., & Lemonides, C. (2007). Spatial abilities in relation to performance in geometry tasks. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education: Working Group 7, Geometrical Thinking* (pp. 1062–1072). ERME.

Panaoura-Maki, G. (2007). *Students' geometric knowledge and skills at the end of the primary education: a comparison of the geometric thinking at the age of primary and secondary education.* Unpublished PhD thesis. University of Cyprus.

Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, *15*, 4–14.

Stylianides, G. (2018). Secondary students' proof constructions in mathematics: the role of written versus oral mode of argument representation. *Review of Education*, *7*(1), 158–182.

Torregrosa, G., & Quesada, H. (2009). Factors limiting configural reasoning in geometric proof. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 477). PME.

Tso, T., & Liang, Y. N. (2002). The study of interrelationship between spatial abilities and Van Hiele levels of thinking in geometry of eight-grade students. *Journal of Taiwan Normal University*, *46*(2), 1–20.

Verschaffel, L., Greer, B., & DeCorte, E. (2000). *Making sense of word problems*. Swets & Zeitlinger.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Winer, M. L., & Battista, M. T. (2022). Investigating students' proof reasoning: Analyzing students' oral proof explanations and their written proofs in high school geometry. *International Electronic Journal of Mathematics Education*, *17*(2), 1–21.

Yavuz, A., Aydin, B., & Avci, M. (2016). The effect of the success in teaching geometry of basic level education mathematics. *European Journal of Education Studies*, *2*(8), 60–71.

Zazkis, D., & Zazkis, R. (2013). Prospective teachers' conceptions of proof comprehension: Revisiting a proof of the Pythagorean theorem. *International Journal of Science and Mathematics Education*, *14*, 777–803.