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RIFLESSIONE E RICERCA

A review of the research in teaching and learning the negative numbers: an "action research" concerning the application of the geometrical model of the number line

Una revisione della ricerca sull'insegnamento e l'apprendimento dei numeri negativi: una "ricerca-azione" sull'applicazione del modello geometrico della linea dei numeri

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Abstract / The notion of negative numbers is one of the most fundamental in mathematics. For many years, they have caused confusion and they have been an object of controversy among great researchers, until the development of symbolic algebra led to their acceptance in the form they have today. When students are first introduced to the concept of negative numbers and the operations between them, various difficulties occur and misconceptions arise, as expected. Their comprehension is one of the most challenging tasks of teaching mathematics due to their complexity and abstract nature. For this purpose, various models have been devised and employed. In the present study we expose some of the epistemological obstacles commonly observed in understanding negative numbers and then exhibit and compare two different models frequently used for introducing the four basic operations between them.

Keywords: negative numbers; geometrical model; number line; action research.

Sunto / La nozione di numeri negativi è una delle più fondamentali della matematica. Per molti anni hanno causato confusione e sono stati oggetto di controversie tra grandi ricercatori, finché lo sviluppo dell'algebra simbolica ha portato alla loro accettazione nella forma che hanno oggi. Come prevedibile, quando gli studenti imparano per la prima volta il concetto di numeri negativi e le operazioni tra di loro, si verificano varie difficoltà e sorgono misconcezioni. La loro comprensione è uno dei compiti più impegnativi dell'insegnamento della matematica a causa della loro complessità e natura astratta. Per questo scopo, sono stati ideati e impiegati vari modelli. Nel presente studio esponiamo alcuni degli ostacoli epistemologici comunemente osservati nella comprensione dei numeri negativi, dopodiché esponiamo e confrontiamo due diversi modelli frequentemente utilizzati per introdurre le quattro operazioni di base tra loro.

Parole chiave: numeri negativi; modello geometrico; linea dei numeri; ricerca-azione.

1 Introduction

The present paper is, in a way, the result of an intensive course in Didactics of Mathematics by a researcher of mathematical education in a group of experienced teachers of secondary school mathematics with high academic qualifications. More specifically, the paper is the result of a close collaboration between the two authors of the paper, that is, a researcher of mathematical education and a teacher from the course team who is also a researcher of mathematics. The main purpose of our paper is to try to find ways to connect the research in mathematics education related to the negative numbers with the didactic practice in school. The paper claims to be an example of connection between research and practice. It consists of a review research devoted to a general study of the issue of teaching negative numbers and an "action research" by a teacher of mathematics. Thus, it is divided in two sections:

– In sec. 2 we present a survey of a large number of publications that are directly or indirectly related to the understanding and learning of negative numbers. We also have included to this first part some articles related to geometrical models of teaching mathematics that could potentially be applied by mathematics teachers to their students.
– In sec. 3 we present a short description of the methodology of the "action research". Then we present the two specific models of teaching the negative numbers that stem from our survey of the research as well as their application to lower secondary school students. Then the "action research" of the mathematics teacher, that consists in the application of the two models by a teacher of mathematics with his students, is presented.

The second section is divided in four subsections. In sec. 2.1, we present some concepts and methods of the didactics of mathematics that have been applied to the understanding, learning and teaching of mathematics such as the historical study of a mathematical concept, the concept of epistemological obstacle, the concept of the didactic contract, the theory of didactic transformation or didactic transposition and the theory of didactic situations.
In sec. 2.2, the above-mentioned concepts and methods are specialized in various studies of historical evolution of the concept of negative numbers, as well as researches of Didactics of Mathematics for the understanding and learning of negative numbers. They are also related to the concept of absolute value, whose historical evolution is closely linked to the historical evolution and understanding of negative numbers.
In sec. 2.3, we present some points of view about the role of representations and the models in mathematics teaching. Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, we insist on the necessity of using a variety of representations in teaching mathematics. We give different examples of researches concerning the use of representations in the teaching and learning of fractions and functions. Furthermore, we argue that the importance of the use of geometrical models in teaching and learning mathematics is very high.
Finally, in sec. 2.4, we present a research about the use of the number line in elementary integer number operations by students of primary school.[1] The phenomenon of compartmentalization is observed because the number line is used as a simple representation rather than as a geometrical model.
The third section is divided in four subsections. In sec. 3.1 we present a brief description of the theory and methodology of action research.
In sec. 3.2, an application of the model of positive and negative charges for teaching negative numbers is presented, as it has been prepared by a mathematics school teacher.
In sec. 3.3 we present an application of the geometrical model of the number line for the teaching

1. The primary school in Cyprus lasts six years and corresponds to the grade 1 to 6.

of the operations between negative numbers as it has been prepared by the same mathematics school teacher.
Finally, in sec. 3.4 we present some results of the comparison of the application of the two models based on a descriptive statistical analysis.

2 Research in mathematics education: the case of teaching and learning the negative numbers

2.1 A brief description of some concepts, methods and theories of didactics of mathematics

A science can only progress if it constantly delimits the scope of its research and defines more accurately the object it intends to study.
Thus, the view that didactics is the scientific study of teaching is not sufficient, as we must consider which aspects of this teaching we will consider and which we will ignore. At this precise point, general pedagogy differs from the pedagogy of a subject.
Didactics of mathematics is an experimental science. Its methodology is inspired by the principles devised for experimental medicine in the 19th century. A trial (experiment) is a prepared teaching situation which depends on a limited number of controlled variables.
Research in didactics requires researchers of various specialities: mathematicians, teachers, psychologists, linguists, statisticians, programmers, historians (of mathematics), and epistemologists. It consequently involves the intervention of multidisciplinary groups. In addition, it requires special methodology and special terminology, without the use of which many didactic phenomena cannot be interpreted. Thus, Didactics of Mathematics has developed a set of concepts and methods that delimit its field of validity (D'Amore & Gagatsis, 1997; Gagatsis & Maier, 1996; Gagatsis & Rogers, 1996). We next refer to some important concepts and methods.
An important method is the didactic transformation or didactic transposition which refers to the processes of transforming scientific knowledge into teaching objects. When a content of scientific knowledge is selected as knowledge for teaching, there is a series of adaptation transformations that will enable it to take its place among the objects of instruction (Chevallard, 1985; Chevallard & Bosch, 2014). Usually, the curriculum is not only determined by the mathematical community. In most countries, these decisions are the result of interactions between mathematicians, teachers, and other social members. They also depend on parents or political choices. School knowledge is the final image of scientific knowledge after the influence of all these factors and after a series of transformations applied to it. The didactic transposition has been studied in a variety of mathematical concepts in different educational systems and, in particular, in the concept of absolute value (Gagatsis & Thomaidis, 1994, 1995, 1996).
The term didactic contract refers to the various teaching situations where it is explicitly defined, at a small part with an implicit manner what is handled by each "partner" in one way or another and for what they are responsible relatively to each other (Brousseau, 1983, 1997). The role of the didactic contract is to regulate the interactions between the teacher and the student in terms of a piece of knowledge. The teacher respects the contract by teaching and giving assessments. He or she tries to make students learn what he or she wants, what they must learn. He or she is the one who knows everything and guides the students to produce their answer by using what they know. The student respects the contract if he or she does the assessments, if he or she learns the lesson. He or she tries to understand what the teacher wants and give the expected answers. For example, he will try to give an answer to a problem proposed by the teacher even if the problem has no solution. However,

this does not imply the comprehension of the mentioned concept. Learning is not based on a blind obedience to the terms of the contract but on the "violation" of the contract.
A characteristic example of the "didactic contract" is related to the absolute value: the students try to find solutions to some equations that are obviously "impossible". For example, the equation:
$||x-5|-12|=-5$ (Gagatsis & Panaoura, 2014; Gagatsis & Thomaidis, 1994).
A key component of the research related to Didactics in France is the theory of the didactic situations. According to Brousseau the theory of didactic situations expresses the conditions under which mathematical knowledge is generated in the context of teaching and allows their study. In this sense a didactic situation is a set of relationships between a teacher and a learner, in which we can recognize a (common) plan of social character that aims to the acquisition of a knowledge by the learner (Brousseau, 1997). Consequently, the teaching of a mathematical notion cannot be done, in this abstract structural form which is met in pure Mathematics. It must be taught in the school environment; it must take the specific form of an interesting problem and be transformed so that its meaning is understood by the students. In this way the problem causes various actions on their part, some formulations of possible solutions, some explanations to his classmates or the teacher, which in a way lead to the "birth" of the concept under teaching. A characteristic example of a didactic situation concerns the "illusion of proportionality" (Modestou & Gagatsis, 2013).

2.2 Historical elements and students' learning obstacles related to the negative numbers

One of the most significant tasks of the Didactics of Mathematics is to identify the obstacles that are opposed to the comprehension and learning of this science. The research indicates that many mathematical concepts have been "marked" by the difficulties of great mathematicians. It is reasonable to assume that many of the difficulties that at some point stopped the most inspired scientists, must still bother our students. The historical study of a mathematical notion is therefore an important method of the Didactics of Mathematics, closely related to the concept of the epistemological obstacle.
In this way the epistemological obstacle becomes an important concept of the Didactics of Mathematics. However, it was observed a great controversy regarding the notion of the epistemological obstacle, in which researchers such as Artigue, Brousseau, Duroux (France), Schubring (Germany), Sierpinska (Canada) etc. took part.
Brousseau (1983) defines obstacles in learning as a collection of errors that are closely related to prior knowledge. This is because such an error arises in the students' initial understanding of a subject, then continues to be repeated so that it is embedded in the system's long-term memory as knowledge.
Brousseau (1983, 1997) states that errors and failures do not have the simplified role that we would like them to play. Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empirist or behaviourist learning theories, but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unsuitable. Errors of this type are not erratic and unexpected, they constitute obstacles. As much in the teacher's functioning as in that of the student, the error is a component of the meaning of the acquired piece of knowledge.
According to the above perceptions, the "obstacle" must have the following characteristics:

- It is a piece of knowledge that works this way in a set of situations and for certain values of the variables of these situations.
- It is a piece of knowledge, which in the effort to be adjusted to other situations or to other values of the variables, will cause specific errors that can be observed and analysed.
- It is a solid knowledge. In situations that escape from the field of its validity, its rejection will cost the students more than one attempt for its adjustment.
- We can only overcome the obstacle in special situations of rejection and this rejection is a component of knowledge.

An obstacle is made apparent by errors, but these errors are not due to chance, they are reproducible, persistent and not necessarily explainable (Brousseau, 1997).
A major problem in understanding negative numbers is the rule of signs. Glaeser (1981) in his article *Epistémologie des nombres relatives* carries out a historical research and shows that even great mathematicians, that have accepted negative numbers and used them in their calculations, encountered problems in comprehending the concept behind them and especially the rule of signs.
The following rule, which was used in British schools, gives an idea of the problem:

> Minus times Minus equals Plus. The reason for that we need not discuss.

Glaeser begun his research after reading the book *Life of Henry Brulard* [autobiography of Stendhal (1783-1843)].

> «What could I imagine when no one could explain to me how it is that a minus times a minus equals a plus (– · – = +)? Finally, I came to what I still say today: the rule "– · – = +" must be true, because it is obvious that, using this rule every time in the calculations, we arrive at results "true and unquestionable". My great misfortune was the following figure:

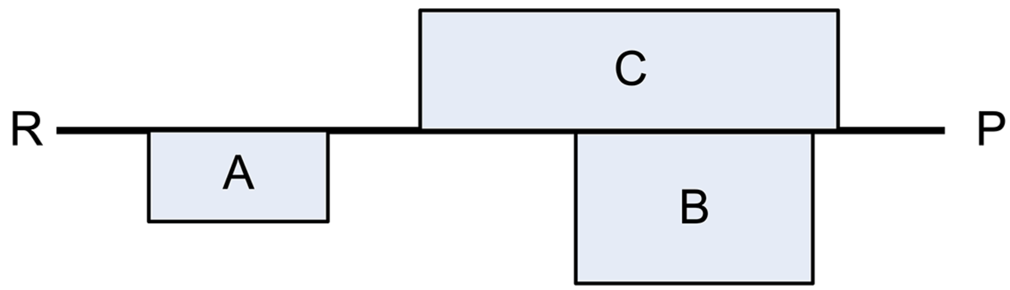

Figure 1. Positive and negative quantities.

> Let RP be the line separating the positive from the negative, everything above it is positive, just as everything below it is negative; how, by taking square B as many times as there are units in square A, can I manage to make it change sides to square C? Suppose the negative quantities are a man's debts, how by multiplying 10'000 francs of debt by 500 francs, will he or can he manage to acquire a fortune of 5'000'000, five million?»

Glaeser begins his study with the texts of Diophantus of Alexandria (late 3rd century AD), to which the "seed" of the rule of signs is generally attributed. Even though this author does not clearly refer to the relevant numbers, at the beginning of his book Arithmetica I, makes an implication to the development of the product of two differences and writes: «Deficiency multiplied by deficiency yields availability; deficiency multiplied by availability yields deficiency».
Based on the study of a large number of mathematical texts related to negative numbers, Glaeser proposes a temporary list of obstacles regarding negative numbers:

1. Inability to manipulate isolated negative quantities.
2. Difficulty in unifying the number line.
   This occurs, for instance, when one insists on the qualitative differences between negative quantities and positive numbers or when one describes the line as a juxtaposition of opposite half-lines

involving heterogeneous symbols or when one refuses to examine the dynamic and static characteristics of the numbers at the same time.
3. The ambiguity of two "zeros".
4. The stagnation at the stage of specific processes (in contrast to the stage of standard processes).
5. The difficulty of dissociating from the specific meaning given to the numbers.
6. The desire for a unifying model: we wish to apply a good additive model, equally valid for explaining the multiplicative model, where this model is functional.

(Glaeser, 1981).

Glaeser's research therefore showed that we had to wait for more than 1500 years for the "rule of signs" to be regarded as something very easy for mathematics. The following table (Table 1) gives a brief overview of the findings of Glaeser's research and gives rise to an intense discussion among researchers for the "nature" of the concept of the epistemological obstacle.

AUTHORS	OBSTACLES 1	2	3	4	5	6
Diophantus	+					
Simon Stevin	+	–	–	–	–	–
René Descartes	+	?	–	?	+	+
Colin McLaurin	+	+	–	–	+	+
Léonard Euler	+	+	+	?	–	–
d'Alembert	+	–	–	–	–	–
Lazare Carnot	+	–	–	–		
Pierre de Laplace	+	+	–	–	+	?
Augustin Cauchy	+	+	–	–	+	?
Herman Hankel	+	+	+	+	+	+

Table 1. Obstacles related to negative numbers.

Bishop et al. (2014) conclude that mathematicians historically grappled with what we describe as three cognitive obstacles related to negative numbers: the lack of a physical, tangible, concrete representation for quantities less than nothing; the problem of removing more than one has; and situations counterintuitive to interpretations of addition and subtraction as joining and separating.

Another major epistemological obstacle related to the notion of negative numbers is the difficulty in understanding them as intellectual objects. Students intuitively understand natural numbers in contrast to negative numbers. The learner necessarily needs to be able to think of natural numbers in a formal manner before he or she can start with negative numbers (Streefland, 1996).

Even though there are differences between the researchers' points of view about the concept of epistemological obstacle, Sierpinska claims that despite these difficulties and regardless of the methodology, the usefulness of epistemological analysis of Mathematics taught at various school levels, seems not discussable, either for the teaching practice or for the writing of textbooks, or as a reference for any type of research in mathematical education.

The researchers have revealed many other difficulties of students related to the negative numbers. For example, Bofferding claims that some of the mistakes and misconceptions that occur in calculations involving negative numbers are related to the way they are denoted. More precisely, after introducing negative numbers the minus sign "–" obtains three meanings, namely: binary, symmetric, and unary (Bofferding, 2014). The binary function refers to any situation where the minus sign shows that the operation is a subtraction (an operation sign). With its symmetric function the minus sign is an indicator of the number being the opposite of its related number, as in the example of 5 and –5. In the unary function the minus sign acts as guidance to the reader that the number is indeed negative, as in –10 being "*negative 10*". According to a research conducted by Bofferding (2010), students most of the times tend to use the binary interpretation of the minus sign, sometimes they use the symmetric interpretation and less often the unary interpretation.

Also, difficulties occur when subtracting integers. Students sometimes have doubts when interpreting situations of the form $a - (-b)$, despite obtaining correct solutions to the problems (Bruno & Martinón, 1999). They struggle to understand that in problems involving subtraction of negative numbers the result is a number greater than the minuend, as they have only been taught how to perform subtraction between positive numbers with the subtrahend always being less than the minuend. In addition, the idea of subtracting a negative number which gives the same result as adding the opposite of the negative number, is difficult for many students to comprehend (Badarudin & Khalid, 2008). When students learn that negative integers exist, they might accept that negative integers are less than zero but argue that –5 is greater than –3 because –3 is further away from zero than –3 is, just as 5 is further away from zero than 3 is (Bofferding, 2014).

Negative numbers today may be found whenever there is a situation with two directions: profit-loss, ahead-behind, deposit-withdrawal, east-west, above sea level-below sea level etc. Operations on these numbers appear when we wish to combine shifts, compare them, transform them, or calculate rates and these models work sufficiently only for addition (Freudenthal, 1983). Most of the models used for teaching negative numbers work for the operations of addition and subtraction. Multiplication and division may require a purely algebraic approach and the concrete models traditionally used would need to be left behind (Williams & Kutscher, 2008).

According to Ekol (2010), students cannot think of an one-to-one correspondence between negative numbers and physical objects, as they have learned to do with natural numbers, and so they try to memorise the rules for performing the operations between them. Understanding negative numbers is in fact, the transition from concrete to abstract mathematics, something for which not all students are ready at the same time period as it requires high abstract thinking ability (Thomaidis, 2009).

The same researcher has studied, in collaboration with Gagatsis, negative numbers in the context of a historical study of the concept of absolute value and has revealed a number of epistemological obstacles related to the notion of natural numbers as a result of measurement (Gagatsis & Thomaidis, 1994, 1995, 1996). In fact Gagatsis and Thomaidis have conducted a historical analysis of the absolute value based on four phases. In the first phase the absolute value appears as an implicit concept, strictly linked to a perception of the relevant number as a common number equipped with a sign. For example, Viète, in his text of 1591, *In artem analyticen isagoge*, introduces a special symbolism for "minus incerto": "A square = B plane", "B plane = A square". In the 17th century negative numbers appear as extensions of "real" numbers to new "fantastic" objects. In the second phase the absolute value plays an explanatory function in the context of algebraic calculus and mainly at the level of the algebra of inequations. The absolute value is displayed as "a number without a sign" or as "a distance from zero". According to Descartes, «The sum of an equation that has some solutions is always

divided by a binome that consists of the unknown quantity reduced by the value of a "true" solution or increased by the value of a "false" solution» (R. Descartes, 1637 – *La Géométrie*, referenced in Gagatsis & Thomaidis, 1995). He continues: «[...] increasing the solutions by a greater amount than any false root will make all the solutions true» (R. Descartes, 1637 – *La Géométrie*, referenced in Gagatsis & Thomaidis, 1995). Let's take, for example, the equation:
$x^3 + 4x^2 + x - 6 = 0$
which according to Descartes' terminology has the "true" (positive solution) root 1 and the "false" (negative solutions) roots 2 and 3. Descartes proposes a transformation: we increase the roots with a quantity greater than a false root e.g. by replacing $y = x + 4$, thus transforming the original equation into an equation with the "true" solutions 5, 2 and 1 (Gagatsis & Thomaidis, 1995).
In the third phase, a close connection with the fundamental conceptual change related to the concept of the number is observed, i.e. the progressive transition from the empirical understanding of number as a mean of measuring quantities, to the abstract concept of number as an element of a mathematical system characterized by the properties of its elements and by the absence of contradiction. For example, for Leibniz $a - b$ means the difference between a and b when a is greater than b, $b - a$ when b is the largest and this absolute value (moles) can be called $| a - b |$ (C. W. Leibniz, *Monitum De Characterisebus Algebraicis*). Finally the fourth phase corresponds to the formalism made necessary by the evolution of complex analysis.
A more recent study by Gagatsis and Panaoura (2014) was based on the historical analysis by Gagatsis and Thomaidis (1994, 1995, 1996) and focused on the similarities of students' personal interpretations of the absolute value with the different types of conceptions that emerged in the historical evolution of the notion. Their analysis was also based on the concepts and methods of Didactics of Mathematics like the didactic transposition, the didactic contract and the epistemological obstacle presented in secc. 2.1 and 2.2. In fact they examined students' performance in problems on absolute value and its relations to didactic obstacles or to epistemological obstacles and the connections between students' conceptions of absolute value (definitions of absolute value) and their performance in solving equations and inequalities incorporatining the notion. The particular study examined students at the second year of upper secondary school (grade 11) and was carried out in Cyprus.[2]
Finally, Elia et al. (2016) attempt to capture the mathematical and cognitive complexity when students deal with the concept of absolute value on the basis of the Mathematical Working Space (MWS) (Kuzniak & Richard, 2014). The MWS includes a network of two planes, a cognitive one and an epistemological one, and this networking is based on three geneses, namely, the semiotic genesis (between representations and visualization of mathematical objects), the instrumental (between artifacts and mathematical construction), and the discursive (between the theoretical reference system and access to mathematical reasoning, argumentation and validation). This last study extends the study by Gagatsis and Panaoura (2014), firstly, by identifying and analyzing indepth students' errors in solving typical and non-typical equations and inequalities incorporating absolute value, by exploring the linkage between these errors and students conceptions (definitions) of absolute value and by discerning more precisely the epistemological or didactic obstacles that intervene in students' mathematical work. Secondly, this analysis takes place within the framework of MWS, with a focus on the discursive genesis and its relations with the semiotic genesis in students' mathematical work. This theoretical framework is complemented with other theoretical approaches based on a historical perspective on the concept of absolute value (Gagatsis & Thomaidis, 1994, 1995, 1996) and the ideas of epistemological and didactic obstacles in mathematical work. A historical view on absolute value can give insight into the epistemological nature and content of the mathematical reference system related to the concept of absolute value in students' personal MWS (workspace in which every student

2. In Cyprus the education system is divided in primary education (grade 1 to 6, ages 6 to 12), lower secondary education (grade 7 to 9, ages 12 to 15), upper secondary education (grade 10 to 12, ages 15 to 18) and higher education.

deals with mathematical items). An analysis of the obstacles students encounter in their mathematical work on absolute value may reveal the sources of students' errors and delays in the genesis of mathematical meaning for the concept and discursive reasoning. The above mentioned authors (Elia et al., 2016) are interested in obstacles either of epistemological nature, which may have commonalities with conceptions in the historical evolution of the concept, or of didactic character, related to the process of the didactic transposition of absolute value and the phenomenon of didactic contract in teaching. An important particularity of this research is that students' errors in exercises involving the concept of absolute value are observed in different educational systems, independently of the students' language and culture (Elia et al., 2016). In fact, a survey was carried out in Turkey following a similar survey in Cyprus in which secondary school students' performance was assessed using a test. Findings showed a discrepancy in the conception of absolute value that was most prevalent in each country, indicating the differences in the reference and suitable MWS between the two countries. For Turkey, the conception of absolute value as distance from zero, which was the most widely used definition, gave a positive support to the solution of problems involving discursive reasoning. This was not the case for Cyprus, in which the most prevalent conception of absolute value was "number without sign". An analysis of the Turkish students' errors revealed a distinction between errors in students' discursive genesis and semiotic genesis, which were a consequence of either didactic or epistemological obstacles that intervened in students' personal MWS. We strongly believe that the MWS model should be applied to the teaching of negative numbers, as well as to the way in which students' perceptions of negative numbers contribute to the formation of the semiotic, the instrumental and the discursive genesis.

2.3 The role of representations and of geometrical models in mathematics and in mathematics teaching: the case of the number line

A characteristic of human intelligence is the use of different types of representations. This characteristic differentiates human intelligence from the intelligence of animals, as well as from artificial intelligence. The element that differentiates human intelligence from that of animals is not just the use of a language but also the use of a variety of systems of representations: verbal language, written language (drawings, paintings, diagrams etc.) (Duval, 2002). Why a variety of semiotic representations? The cost of processing, the limited representation capabilities and the necessity of at least two representations for the understanding of a mathematical concept, are some of the reasons. Thus, the representational systems are fundamental for conceptual learning and determine to a significant extent what is learnt. Recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Elia et al., 2008; Goldin, 2001; Lesh et al., 1987) and allow students to see rich relationships. Thus, the different types of external representations in teaching and learning mathematics seem to have become widely acknowledged by the mathematics education community (National Council of Teachers of Mathematics, 2000). Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002, 2006).
The necessity of using a variety of representations or models in supporting and assessing students' constructions of fractions is stressed by a number of studies (Deliyianni & Gagatsis, 2013; Deliyianni et al., 2016; Gagatsis et al., 2011; Gagatsis et al., 2016).
Moreover, the concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decades. A vast number of studies have underlined the importance of different representations and the translation between them in understanding the concept of functions (Elia et al., 2007; Elia et al., 2008; Even, 1998; Gagatsis & Shiakalli, 2004; Panaoura et al., 2017).

On the other hand, the importance of the use of geometrical models in teaching and learning mathematics is very high. Patronis gives an important contribution on the nature and the use of geometrical models in mathematics teaching. He uses a large scientific bibliography about mathematical models and in particular some of the work of Thom. According to him, mathematization (or, for some people, the process of "model building") can be viewed as consisting of two human activities of equal importance, none of which could be ignored without unhappy consequences. The first activity, which is usually more stressed when speaking of mathematical models in scientific practice, is prediction. The second one is explanation. Prediction is not explanation, as Thom emphasizes (Thom, 1982, 1991).
Each of the above two human activities intend to a different aim of scientific research: prediction intends to successful action, while explanation intends to (better) understanding. Following Thom, we may consider these two aims as complementary to each other: in order to act successfully, one needs first to understand and conversely, in order to obtain a better understanding of some phenomena – or, in order to obtain a "global" understanding – one need first a kind of "local" action and representation, so that a "global" picture then results by recollecting the "local" ones. However, by focusing on one aim, the human mind tends to forget the other, and this complementarity is usually conceived as an opposition (Gagatsis & Patronis, 2001).
In proceeding for "local" action to "global" understanding, geometrization, the act of constructing "geometrical models", stands at an intermediate position. On the one hand, geometrical representation of a system provides a "global" (or "holistic") picture of it, thus helping us to grasp its meaning and functions at once. This "immediate" understanding can be contrasted to the "linear" process of reading a text. On the other hand, a "geometrical model" in the sense of Thom connects the (local) singularities of the system with its (global) structure and explains the structure in terms of the singularities (Thom, 1982, 1991).
The concept of geometrical model seems not to have been systematically explored by philosophers and epistemologists (with the possible exception of Bachelard, 1986).
In a previous study, Gagatsis and Patronis (1990) examined the role of geometrical models in mathematics education, in connection with the development of a process of reflective thinking, which may be observed in the activity of students, of teachers and of mathematicians in their own research. The main conclusion in that study was that young children develop series of similar or continuously deformed geometrical shapes of triangles, rectangles and squares, which can be approximately identified to continuous paths in a space of polytopes. Such series of shapes had been used to teaching and explaining the idea of transformations in geometry, as well as in representing geometrically the algebraic relations $x + y = s$, $x \cdot y = p$.
Emma Castelnuovo justified in a way these teaching practices, since explanatory geometrical models used by teachers appeared implicitly in children's own thought perception and action. What in fact means this quasi-empirical conclusion, in view of the preceding theoretical remarks, is that, at least some of the geometrical "universes", which have been used by scientists and teachers in models of various "systems" could also be considered as models of human or cultural activity in its first informal appearance (Gagatsis & Patronis, 1990).
The above authors suggest a definition of the geometrical model:

> «We shall say that a collection S of points, lines or other figures in n-dimensional Euclidean space, representing a system Σ of objects or a situation or process, is a (theoretical) geometrical model of Σ, if the intrinsic geometric properties of the elements of S are all relevant in this representation, i.e. they correspond to properties of the system Σ. If this condition is satisfied only for the topological properties of lines or figures in S then we shall speak of a geometrical model in the wide (or topological) sense».
>
> (Gagatsis & Patronis, 1990)

According to Fischbein, a model must have a generating character, that is, it must generate (produce) and represent an unlimited number of properties, starting from a limited number of elements and rules for combining them. In addition, a model must have a heuristic character, that is, to lead us easily, and regardless of the initial system it represents, to new information about that system (Fischbein, 1972).

2.3.1 What is the difference between a representation and a geometrical model?

It is important to distinguish between the terms "representation" and "model": a model is a way of representation (like language), but a representation is not a model, unless it has a generating and heuristic character. The representation may contain more elements or omit some elements of the whole to be represented. While in the geometrical model all its elements give some information about the whole to be represented: it has generating and heuristic character.
Based on the previous and on the views of Patronis we give the following definition for the geometrical model of the number line:
Let Σ be the ring of integers or the field of rational numbers – or a field, which contains the field of rational numbers – with the following operations: addition, subtraction, multiplication and division.
Let S be a numbered line, that is, a line with a distinct set of points, which correspond to integers or rational numbers. S is a geometrical model for Σ.
There is an isomorphism between Σ and S, so that the operations between the numbers, belonging to Σ, correspond to the operations between directed line segments or the operations between numbers and line segments.
Example: the product of a number and a directed line segment.

2.4 A research about the use of the number line in elementary integer number operations by students of primary school

Many elementary mathematics methods texts recommend the use of number line diagrams for teaching addition and subtraction of integers. Elementary curriculum planners include the ability to perform simple calculations involving addition and subtraction of integers with the aid of a number line diagram amongst their objectives. Some mathematics educationists relate the ability to use a number line diagram with the understanding of elementary integer operations. On the contrary, some others reject this idea: for example, Hart (1981, in Gagatsis et al., 2003) argue that the number line model should be abandoned. Liebeck (1984, in Gagatsis et al., 2003) claims that the number line is not a useful visual aid for helping younger children to add and subtract integers. On the other hand, Ernest (1985, in Gagatsis et al., 2003) remarks that there can be a mismatch between students' understanding of addition of integers and their understanding of the number line used for this operation (Gagatsis et al., 2003).
Gagatsis et al. (2003), motivated by the criticisms about the use of the number line in order to perform simple calculations involving addition and subtraction between integers, conducted a quantitative survey on elementary school students. They gave to students four questionnaires on integers. Questionnaire A included calculations with addition and subtraction of integers in symbolic form. Questionnaire B included similar or identical calculations with addition and subtraction of integers in symbolic form, but also enabled students to use the number line. Questionnaire C included similar or identical calculations with addition and subtraction of integers in symbolic form but students were obliged to use the number line and show the result and process on it. Finally, questionnaire D contained diagrams of a number line on which integer operations were portrayed with arcs and students had to find the numerical relationships of the operations as well as their results. In the following diagrams we observe the results.

2.4.1 Results in questionnaires A and B

We observe that in all the cases the score in the questions of questionnaire B is better than the score in the questions of questionnaire A (exception A1-B1).

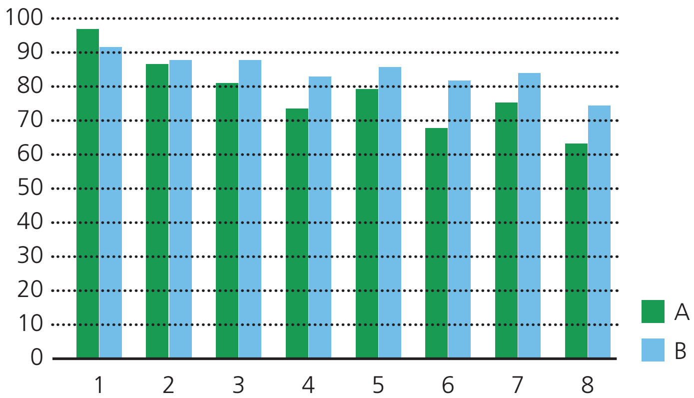

Figure 2. Comparison between the scores of questionnaires A and B.

2.4.2 Results in questionnaires C and D

There is no significant difference between the scores in the questions of questionnaires C and D.

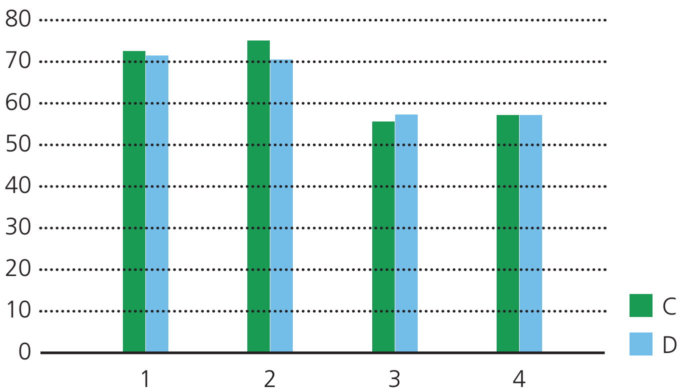

Figure 3. Comparison between the scores of questionnaires C and D.

2.4.3 Application of the implicative statistical analysis of Gras

Gras developed a very useful and effective statistical method for teaching and learning issues, and even more, which through the CHIC software, gives implicative diagrams, i.e., hierarchical relationships between variables (Gras, 1979; Gras & Couturier, 2013). When an arrow connects two variables, it means that the success of students in one of the variables (problems, questions etc.) implies success in the other. The importance of such implications is considerable not only for teaching and learning mathematics but also for other subjects. In addition, the use of this method is widespread in various other sciences such as psychology, social sciences, medicine etc.

We applied this method to the students' responses to questionnaires A, B, C, D, and as we can see in the below diagram, there are implications only between the tasks of questionnaire C and questionnaire D. This means that students who have correctly solved the exercises in questionnaires C and D will not achieve correct answers in some of the exercises in questionnaires A and B. In fact, we observe the phenomenon of compartmentalization and the reason for this is the existence of the number line on which students must work in C and D. That is, the young students use the number line as a simple representation and not as a geometrical model. This phenomenon of compartmentalization has also been observed in other researches (Elia et al., 2005; Gagatsis et al., 2011; Shiakalli & Gagatsis, 2005a, 2005b).

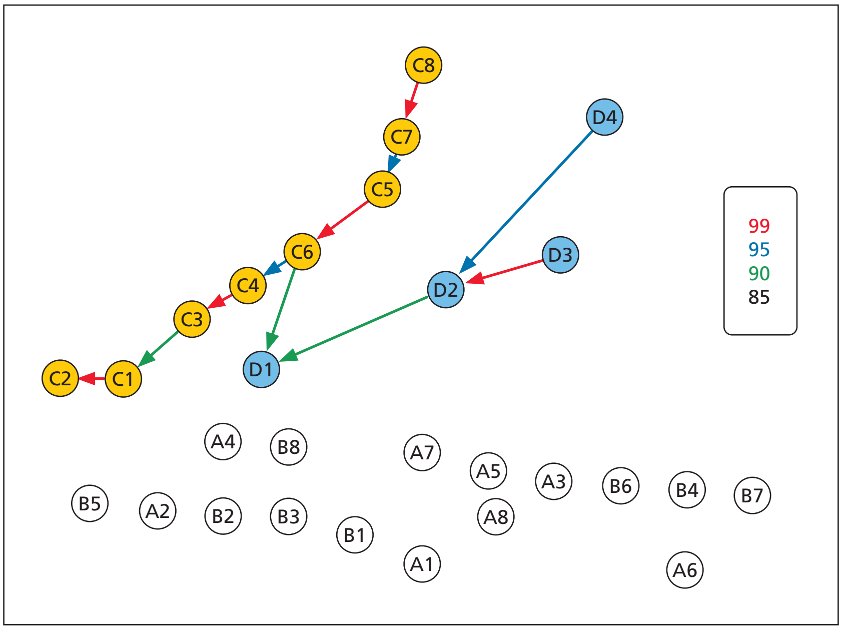

Figure 4. Implicative statistical diagram of questionnaires A, B, C and D.

Finally, we have to mention that the same research has been repeated in Greece and in Italy and the same phenomenon of compartmentalization has also been observed despite the differences between the educational systems, the language and the culture of the three countries (Gagatsis et al., 2004). Based on the results of the above research we can understand the views of some researchers that the number line is not a useful visual aid for helping younger children to add and subtract integer numbers. The number line is a geometric model that helps students develop a variety of skills which can be applied not only to the operations between integers but also in other concepts.

3 A comparison of two models for teaching the negative numbers: an action research

3.1 Some elements of the methodology of action research

Action research is the process of studying a real school problem or situation. In other words, action research is a method of systematic enquiry that teachers undertake as researchers of their own practice (Efron & Ravid, 2020; Mertler, 2009). Teacher empowerment can be facilitated with action research. In fact, action research can be used as a replacement for traditional inservices to enhance teachers' professional growth and development. Moreover, the goal of action research is to improve one's teaching practice or to enhance the functioning of a school. Johnson propose some essential steps of action research (Johnson, 2012). The first step is the definition of a question or area of study; in our case it is the teaching and learning of the negative numbers. The second step is the teachers' decision about the method for collecting data. In fact, different methods could be used: the observation of individuals or groups; the use of audio and video tape recording; the use of structured or semi-structured interviews; the distribution of surveys or questionnaires etc. The third step is the collection and the analysis of data: there exist different methods of analysis of qualitative or quantitative data. The fourth step is the description of the different ways of the use and application of the findings (Johnson, 2012). A review of the relevant literature related to the topic or question of research might also be included in action research.

In our case, the use of models has been a central focus for teaching negative numbers and has been studied by various researchers. The review of the relative literature has been presented in the first part of the paper. According to some studies (Battista, 1983; Liebeck, 1990; Stephan & Akyuz, 2012), there are two models for integer instruction, namely the one with counters or charges of two types and the one with number lines. Concerning the methodology, in our action research we have used two questionnaires that correspond to the above two models for teaching the negative numbers. We have applied the descriptive statistics in order to analyze our data.

3.2 The model of positive and negative charges for teaching negative numbers

In the first model we use a specific representation for the two types of charges, each of them corresponding to either a positive or a negative unit. A positive charge is represented by the symbol "+" in a circle. Similarly, a negative charge is represented by the symbol "–" in a circle. The operations between numbers are associated with the manipulation of the charges and the calculation of the total charge. Both addition and subtraction are based on the use of the identity element. The identity element of an operation is one such when combined with another number under that operation leaves the number unchanged. For addition and subtraction this is the number zero. Students must understand that two charges of different type correspond to the number zero, that translates into the calculation (+1) + (–1) = 0. This can be extended as follows: adding any number of charges which can be put into pairs of opposite types does not change the total charge that we initially had, so $v + 0 = v$. Thus, we are allowed to draw or cross off any number of such pairs for ease of calculations.

When adding we draw the charges that represent the two numbers to be added, using the appropriate symbols depending on whether they are positive or negative. We then find all the possible pairs of opposite charges and cross them off (as their value is zero) and finally we count the remaining charges (that must be of the same type). For instance, in order to perform the addition 4 + 3 we draw four positive charges and then three more positive charges. Therefore, in total we have seven positive charges, and the sum must be seven. For the addition of two negative numbers, we work in a similar manner. In order to add two numbers with different signs we use the property of zero as the identity element. For instance, to perform the calculation (–4) + (+3) we first draw four negative charges and then three positive charges. We observe that we have three pairs of opposite charges, which we do not take into account. Thus, we end up with one negative charge (Table 2).

Operation	Symbolic representation	Answer
(–4) + (+3)	⊖⊖⊖⊖ ⊕⊕⊕	(–4) + (+3) = –1

Table 2. Sum (–4) + (+3).

The operation of subtraction is performed as follows: we draw the charges corresponding to the number to subtract from and then we cross off the charges that correspond to the number to be subtracted. In the cases where there are not enough charges to be crossed off we draw additional charges. For every additional charge we draw another charge of the opposite type as well, so that the initial charge stays the same. Finally, we count the remaining charges. For instance, in order to perform the subtraction (–4) – (+3), we start by drawing four negative charges. We then have to cross off three positive charges, that do not exist. To find the difference we draw three positive and three negative charges. After that we cross off the three positive charges and so we eventually end up with seven negative charges (Table 3).

Operation	Symbolic representation	Answer
(–4) – (+3)	$\ominus\ominus\ominus\ominus\ominus\ominus\ominus$ $\cancel{\oplus}\cancel{\oplus}\cancel{\oplus}$	(–4) – (+3) = –7

Table 3. Difference (–4) – (+3).

To understand multiplication, students only need to realize that this operation is equivalent to repeated addition. For example, to do the multiplication 3 · (–2) we have to draw two negative charges three times, thus in total we have six negative charges. The hardest case with multiplication is the one in which we have two negative factors. We must draw pairs of opposite charges (enough to cross off the negative charges) and then cross of as many negative charges as the value of the modulus of the second factor for as many times as the modulus of the first factor. In other words, we cross off as many negative charges as the product of the moduli of the two factors. Finally, we count the remaining charges (we do not take into account any pairs of opposite charges). Consequently, the product of two negative numbers is always positive. For example, to perform the calculation (–3) · (–2) we draw pairs of opposite charges and then cross off two negative charges, repeating three times. Therefore, we cross off six negative charges in total and six positive charges are left (and possibly some pairs of opposite charges) which implies that the product is six.

When dividing we try to obtain as many charges as the dividend by repeatedly drawing or crossing off charges. In the cases where the divisor is a positive number, we draw groups of charges with cardinality equal to the divisor. In these cases, the sign of the result depends on the sign of the divisor. If we have for instance the division (+6) : (+2), we draw pairs of positive charges until we obtain six charges. We need to draw three such pairs and thus the quotient is +3. If we now have the division (–6) : (+2), we draw pairs of negative charges until we obtain six of them. We need to repeat this process three times and so the quotient is –3. In the cases where the divisor is negative the process changes. We basically try to create and cross off as many equivalent groups of charges as the divisor, so that we end up with as many charges as the value of the dividend. If we have the division, (+6) : (–2), we cross off two equivalent groups of charges so that we end up with 6 positive charges. Thus, we initially draw six pairs of opposite charges and cross of two triples of negative charges. Hence the quotient is –3, since we cross off triples of negative charges. Similarly, we perform the division (–6) : (–2). We draw six pairs of opposite charges and cross off two triples of positive charges. Therefore, the quotient is +3, as we cross off triples of positive charges.

3.3 The application of the number line for the teaching of the operations between negative numbers

Based on the review on one hand of the research on the mathematicians' and the students' obstacles related to the negative numbers and on the other hand on the advantages of the use of geometrical models in the teaching of mathematics, we propose the model of the number line for the teaching of negative numbers.

This is a line that contains all real numbers and is composed of three basic parts. In the middle, we have the origin usually plotted with the number zero, positive numbers are plotted to the right of the origin while negative numbers are plotted to the left of the origin. Arrows are put on the ends of the horizontal line to show that the line continues to infinity. Vectors are used for the representation of numbers. For positive numbers we use vectors directed to the right while for negative numbers we use vectors directed to the left. The length of a vector corresponds to the absolute value of the number that it represents. For instance, the number 4 is represented by a vector of length four directed to the right, while the number –4 is represented by a vector of length four directed to the left.

In this model addition is performed as follows: starting from zero we draw a vector corresponding to the first number. Next, starting from the end point of this vector we draw another vector corresponding to the second number. The sum of the two numbers is represented by a vector which starts from the start of the first vector (that is zero) and extends to the end of the second vector. For example, if we have to do the calculation (+2) + (+3), we first draw a vector extending from zero to 2 and then starting from 2 we draw a vector of length three, directed to the right which therefore extends to 5. The sum of the two numbers is represented by a vector which starts from zero and extends to 5 and thus represents the number 5 (Figure 5). In a similar manner, we can add two negative numbers, with the difference that the vectors are both directed to the left this time. If we have to add two numbers with opposite signs, say for instance we have the calculation (+2) + (–3), we work as follows: we first draw a vector from zero to 2 and then starting from 2 we draw a vector of length three and direction to the left which therefore ends at –1. The sum of the two numbers is represented by a vector that starts from zero and extends to –1 and represents the number –1 (Figure 6).

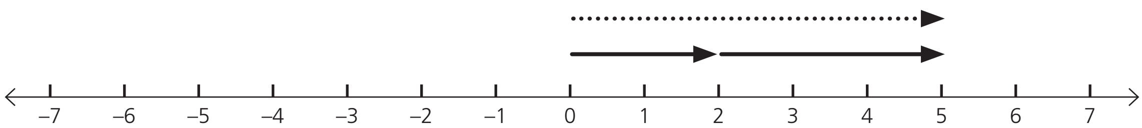

Figure 5. Sum (+2) + (+3).

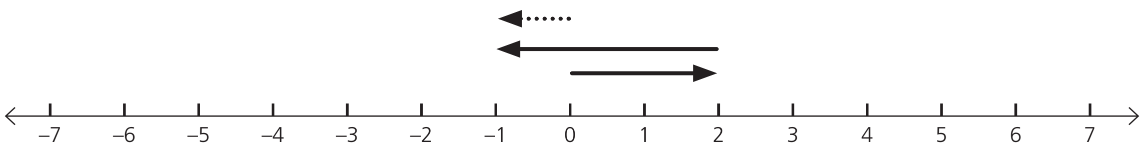

Figure 6. Sum (+2) + (–3).

Subtraction is performed similarly, as illustrated with the following two differences: the second vector is placed so that its end point is the same as that of the first vector and the difference of the two numbers is represented by a vector which begins from the starting point of the first vector and ends at the starting point of the second vector.

For example, if we have the calculation (+7) – (+3) we work as follows: we draw a vector beginning from zero and ending at 7. Next, we draw one more vector ending at 7 as well, having a length of 3 and direction to the right, thus it must have its starting point at 4. The difference between the two given numbers is represented by a vector starting from zero and extending to 4, so it is the positive number 4 (**Figure 7**). In the case of two numbers with opposite signs, such as for example in the subtraction (–7) – (+3) we do the following: we draw a vector starting from zero and extending to –7. After that we draw one more vector ending at –7 as well and having a length of three. The second vector is directed to the right, so it has to start from –10. The difference of the two numbers corresponds to a vector starting from zero and extending to –10, so it is the negative number –10 (**Figure 8**). Finally, if we have to subtract a negative number as for example in the calculation (+7) – (–3) we work in the same manner: we draw a vector starting from zero and extending to 7. Afterwards we draw another vector ending at 7 and having a length of 3. The second vector is directed to the left and thus has to start from 10. The difference between the two numbers corresponds to a vector starting from zero and extending to 10, so it represents the positive number 10 (**Figure 9**).

Figure 7. Difference (+7) – (+3).

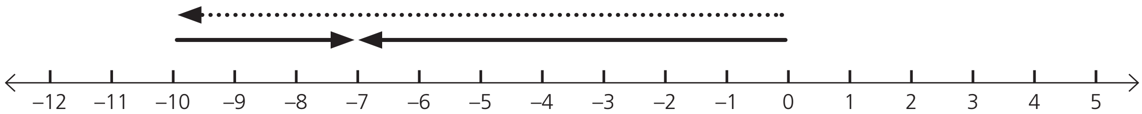

Figure 8. Difference (–7) – (+3).

Figure 9. Difference (+7) – (–3).

Multiplication between positive numbers is achieved by repeated addition. Multiplication of a positive number by a negative number is performed likewise. For example, for the multiplication (+2) · (–3) we follow the next steps: we draw a vector with starting point at zero and ending point at –3 and then one more vector starting at –3, with a length of 3 which has to extend up to –6 and thus the product is –6 (**Figure 10**). In the case where the first factor is a negative number, the operation of multiplication implies repeated subtraction from zero, that is the second factor is subtracted from zero as many times as indicated by the modulus of the first factor. If for instance, we have the multiplication (–2) · (–3) we do the following: we draw two consecutive vectors of length 3 and directed to the left (representing the number –3). Consequently, these vectors have to start at 6. Thus, the product of the two numbers is represented by a vector having its starting point at zero and its ending point (at the starting point of the two vectors) at 6 (**Figure 11**). Hence, the product of –2 and –3 is 6.

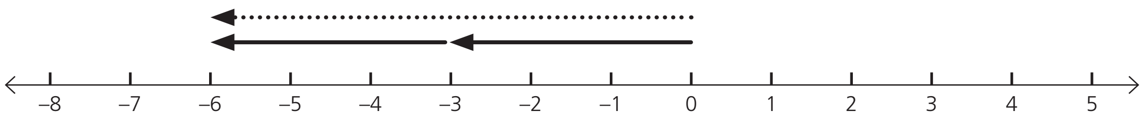

Figure 10. Product (+2) · (–3).

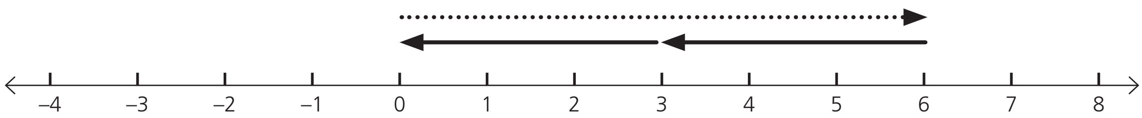

Figure 11. Product (–2) · (–3).

Division is achieved either by repeated addition or by repeated subtraction of the divisor from zero as many times required to obtain the dividend. In order to perform the division (+10) : (+2) we draw a vector from zero to 2, then one vector from 2 to 4 and repeat this process three more times until we reach 10. Since we must draw 5 consecutive vectors of length 2 to reach 10, the quotient of this division is 5 (Figure 12). For the division (–10) : (+2) we repeatedly subtract from zero: we draw a vector ending at zero and starting from –2 (as it has a length of two and is directed to the right), then one more vector ending at –2 and starting at –4 and repeat this process three more times until we reach –10. Since we must subtract from zero five vectors of length 2 to reach –10, the quotient of this division is –5 (Figure 13).

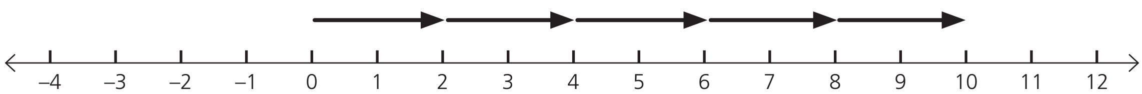

Figure 12. Quotient (+10) : (+2).

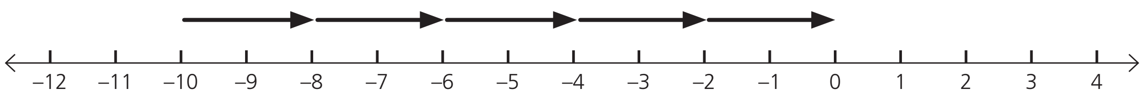

Figure 13. Quotient (–10) : (+2).

3.4 Results of the comparison of the application of the two models

The purpose of this study is to compare the model of positive and negative charges and the model of the number line, in terms of their effectiveness in teaching the operations of addition and subtraction of integers. The research involved 18 students between 12 and 13 years old, who were divided into two homogeneous groups. The first group was given instructions on how to apply the positive and negative charges model along with some examples. The second group was given instructions and examples on the number line model. Each group was given a different worksheet. In the two worksheets, the groups were asked to perform the same operations each using the method explained to them.

More precisely, the two worksheets included two activities (Activity 1 and Activity 2) and each of them had four sub-questions (a, b, c, d). Questions 1a and 1b concerned the addition between two positive and two negative numbers, respectively. Questions 1c and 1d concerned the addition of one positive and one negative number. In 1c the absolute value of the positive number was greater than that of the negative number and in 1d the opposite was true. Questions 2a and 2b concerned subtraction between positive numbers. In 2a the minuend was greater than the subtrahend and in 2b the reverse. Question 2c concerned the subtraction between two numbers with opposite signs with the subtrahend being negative. Question 2d concerned subtraction between negative numbers. Below (Table 4) are the success rates per question in each worksheet.

Question	Operation	Positive and negative charges model	Number line model
1a	(+5) + (+1)	78%	78%
1b	(–2) + (–3)	78%	67%
1c	(+5) + (–1)	56%	44%
1d	(+2) + (–6)	33%	44%
2a	(+6) – (+4)	44%	33%
2b	(+1) – (+4)	22%	11%
2c	(+2) – (–4)	11%	0%
2d	(–1) – (–5)	11%	0%

Table 4. Success rates per question in the two worksheets.

We observe that the success rates in question 1a are the same for both groups. A small difference in success rates is observed in questions 1b, 1c and 1d. The charges model prevails in questions 1b and 1c and the number line model has a higher percentage in question 1d. There is a difference in the success rates in questions 2a, 2b, 2c and 2d with the charges model having the highest percentages in all.

From the above it follows that the number line model is less effective for the operation of subtraction compared to the charges model. This may be due to the fact that there are two differences in the procedure followed in subtraction compared to that followed in addition (the arrow corresponding to the subtrahend is placed so that it ends at the same point as the arrow corresponding to the minuend instead of starting from where the arrow corresponding to the minuend ended and the arrow corresponding to the difference starts at the beginning of the first arrow and ends at the beginning of the second as opposed to that of the sum starting at the beginning of the first arrow and ending at the end of the second). These differences are likely to be confusing as students attempt to apply to subtraction the same procedure they used for addition.

Success rates in questions 2a-2d involving subtraction are generally lower than success rates in questions 1a-1d involving addition for both models. Hence, we conclude that it is more difficult for students to understand subtraction. Students encountered a special difficulty in the cases where the minuend was a negative number.

Various researchers have attempted the comparison of these two models. According to Bofferding (2014) the two primary models for teaching negative numbers are the model of positive and negative charges and the number line model. The model of positive and negative charges is a model of the first type as its functionality is based on the fact that a positive and a negative charge cancel each other. The model of positive and negative charges involves using the additive inverse principle, which could aid partial understanding of the unary minus sign, but it does not emphasize order; and in some cases, subtracting involves both addition and subtraction, potentially confusing the binary meaning of the minus sign. On the other hand, the number line model highlights the order of negative numbers compared to positive numbers, and number values can be interpreted as distances from zero in opposite directions, providing meaning for the unary meaning of the minus sign and directed magnitudes. Despite the advantages of the number line model, this method is not helpful if the learner does not have a clear understanding of an abstract number line. In addition, a common mistake of students when applying this model is that they are counting the numbers themselves rather than the spaces between the numbers. Freudenthal (1983) pointed out two more disadvantages of the number line model. The first fault of the model is the didactical asymmetry between positive and negative numbers. The positive numbers are more concrete in the sense of greater originality; so, one can operate with them; the negative numbers are secondary, introduced as results of operations, which formerly were impossible, fit to be operated on if need be, but unfit to have operations performed with them. Secondly, in this model a point *a* on the number line is at the same time interpreted as an arrow from 0 to *a* and there are textbook authors who tacitly switch from that interpretation to the view of arrows as numbers and by this way suggest more than the first interpretation can yield.

4 Conclusions

The comprehension of negative numbers and the operations between them is a rather demanding task, because of their nature. Difficulties arise not only in conceiving their existence but also in understanding how to use them. That is, common mistakes occur in both calculations involving negative numbers and conceptualizing and making sense of them and their operations. It is clear that the small number of pupils to whom the model of positive and negative charges and the geometrical model of the number line were applied, does not allow us to use the implicative statistical analysis, as was done in the survey of primary school students.

Though, with the appropriate teaching approach, in particular the approach based on the geometrical model of the number line, we can overcome these persistent obstacles and misconceptions that have been encountered by both great mathematicians and students. In fact, the geometrical model of the number line has a generating character, that is, it produces and represents an unlimited number of properties, starting from a limited number of elements and rules for combining them. In addition, this model obviously has a heuristic character, that is, it easily leads us to new information about that system, regardless of the initial system it represents.

Nevertheless, the geometric model of the number line seems to us to be more suitable for more advanced students. The results of the application of the number line in simple tasks on addition and subtraction of integer numbers in primary school students supports this point of view. Moreover, this model, which is ultimately a special case of an affine line model with vector calculation, should be more generally integrated into an affine plane with vectors. Even though the present "research in action" is based to a simple methodology and the descriptive statistical presentation of the results, we believe that it is a concrete good example of a relation between the "research" in mathematics education and the "practice" in schools. We argue that action research can be used to bridge the gap between mathematics education research and teaching practice in mathematics.

Finally, we strongly believe that the model of Mathematical Working Space (MWS) should be applied to the teaching of negative numbers as well as to the way in which students' perceptions of negative numbers contribute to the formation of the semiotic, the instrumental and the discursive geneses.

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